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A NEW DECONSTRUCTIVE LOGIC: LINEAR LOGIC

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Abstract. The main concern of this paper is the design of a noetherian and confluent normalization for $\mathbf{LK}^2$ (that is, classical second order predicate logic presented as a sequent calculus).

The method we present is powerful: since it allows us to recover as fragments formalisms as seemingly different as Girard's **LC** and Parigot's $\lambda\mu$, **FD** ([10, 12, 32, 36]), delineates other viable systems as well, and gives means to extend the Krivine/Leivant paradigm of 'programming-with-proofs' ([26, 27]) to classical logic; it is painless: since we reduce strong normalization and confluence to the same properties for linear logic (for non-additive proof nets, to be precise) using appropriate embeddings (so-called decorations); it is unifying: it organizes known solutions in a simple pattern that makes apparent the how and why of their making.

A comparison of our method to that of embedding **LK** into **LJ** (intuitionistic sequent calculus) brings to the fore the latter's defects for these 'deconstructive purposes'.

§1. Introduction. It has long been thought that classical logic cannot be put to use for computational purposes, as in the 'programming with proofs'-paradigm, based upon the so-called Curry-Howard correspondence ([17]) between proofs in intuitionistic logic and λ-terms (providing, on the theoretical side, a mathematical foundation for functional programming languages, and, from a more practical point of view, a *method* of programming which ensures correctness of programs, cf. [26, 27]). "Classical logic", it was said, "is non-constructive", also in this, highly specific, sense. This, however, turned out to be no more than a, somewhat longstanding, prejudice (as witnessed by our 'programming theorem' in the last section), nurtured by the fact that nothing better than a mere cut-elimination (weak normalization) theorem was to be found in the proof theory of classical logic. That is before the attempts of the last few years.[1]

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[1] Some exceptions are Prawitz's strong normalization for classical natural deduction, and Dragalin's 'strong form of the normalization theorem' for **LK**, in [6]. Note that Dragalin's result only holds with respect to a specific *strategy* for eliminating the cuts: a cut, one of whose premises ends in a cut, while the other one ends in either a cut or a logical introduction of the cutformula, is not considered a redex. Also, there is no reasonable possibility for a denotational semantics, no second order, et cetera. (For those familiar with Dragalin's proof a remark which will become clear later on: the difference does certainly *not* reside in the, from our point of view, somewhat unusual formulations of the logical rules in his calculus, i.e., we can prove 'tq-SN' for Dragalin's **LK** by 'simulation'.) Dragalin's proof was adapted to suit the two sided sequent calculus for **LL** by Roorda (see [43]), but observe that the resulting **LL**-normalization theorem is essentially different from Girard's strong normalization theorem for proof nets.

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Right. But what would one gain, were it possible to obtain a Curry-Howard type correspondence between some 'abstract, rudimentary' programming language and proofs in *classical* logic?

As is well known, by a result dating back to Kreisel ([21]), the interest can not lie in the access to new representable functions, as all functions representable, in, say, classical second order predicate calculus, are already so in the *intuitionistic* second order system (cf. also Chapter 10 of [22, 23]). However, one will have access to other *proofs* of theorems of the form $\forall x\ (N(x) \to N(f(x)))$, and thus to other, and maybe more efficient, algorithms. Apart from the obvious theoretical interest, it is in this potential application that lies the 'classical logic constructivization'-quest's grail. But the question of new classical 'algorithms' only makes sense given an evaluation scheme, which in turn presupposes a syntax equipped with a normalization.

Several such proof systems for classical logic have recently been proposed, notably **LC** (*Logique Classique*, [10]) by Jean-Yves Girard and **FD** (*Free Deduction*, [32]) by Michel Parigot. At first sight these do not seem to have much more in common than the bare fact of being both complete for provability. Girard's **LC** is distinguished by coming equipped with a *denotational semantics*, while Parigot developed an extension of the λ-calculus, the so-called $\lambda\mu$-calculus, whose terms correspond to proofs in a (complete) 'fragment' of **FD** (the 'classical natural deduction'[2] (**CND**) of [33]), thus extending the Curry-Howard isomorphism between λ-terms and proofs in intuitionistic natural deduction to $\lambda\mu$-terms and proofs in classical natural deduction.

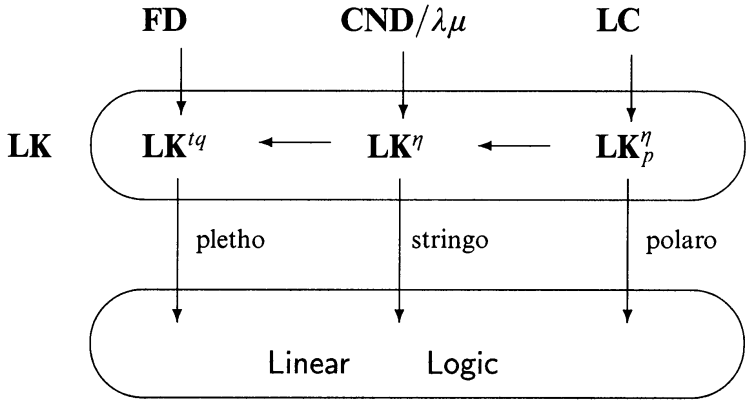
Our own approach is based upon earlier work on the linear decoration of derivations in classical sequent calculus (see [18, 40, 4]). There we observed that restrictions on classical derivations, comparable to those in Girard's **LC**, are induced by certain modal translations of classical into linear logic. Thus we defined calculi (we called them **LKT** and **LKQ**) that are complete for classical provability, and moreover interpretable as *fragments* of linear logic, hence inheriting linear logic's computational properties. Also we found that, e.g., **LKT** on the one hand properly contained a *fragment* (the negative one) of **LC**, on the other hand could *interpret* Parigot's **CND**: the seemingly very different solutions to the problem of a 'computational' classical logic apparently touch common ground in a 'linear image' of classical sequent proofs.

This suggests the possibility of using embeddings of classical into linear logic as a classifying and clarifying tool: by reformulating the problem as that of identifying the possible ways in which one somehow faithfully can interpret classical sequent derivations as proofs in linear logic we (a) conveniently delimit the 'space' in which to look for constructivizations of classical logic, and (b) know that, due to linear logic's proof theoretical properties, success is guaranteed whenever such an interpretation is found.

Our point of departure will be a most general and symmetric version of classical sequent-calculus, namely 'all-style' **LK**, introduced in Section 2.2. Surely if we can devise solutions within such a big system, we will be able in the aftermath to specialize them to fragments. (To quote a renowned French mathematician: "Tant qu'on n'a rien fait, pas la peine de refréner ses ambitions".) We then turn to the formulation and proof of the one basic result we will use about the normalization

[2]Not to be confused with Prawitz's system of the same name.

of **LL** proof nets. This ends Section 2. In Section 3 we discuss 'pathologies' of **LK** with respect to normalization and show how linear logic unveils precisely these problems, and moreover suggests possibly adequate treatments. In Sections 4, 5 and 6 we propose three different 'therapies', each of which is shown to correspond to a different kind of embedding of **LK** into **LL**. The following diagram, very, very roughly, summarizes the content.



In Section 8 we address the question whether for our analysis we could have used **LJ** as a substitute for **LL** and conclude that "well, not really". And in the final section we show that **LK**, augmented with equational reasoning, like **LJ**, can be used as a programming language.

§2. Notations and other prerequisites. Let us introduce and explain some of the terminology that we will use with respect to our main objects of study, *derivations in sequent calculi.*

2.1. General. *Sequents* are denoted by $\Gamma \Longrightarrow \Delta$, where $\Longrightarrow$ is the *entailment sign* of the calculus, and Γ, Δ are finite *multisets* of formulas, i.e., sets with multiplicities, or (equivalently) lists modulo the order in which the entries are given. In particular *exchange* is implicit: $\{E, W, T, E\}$ and $\{T, E, E, W\}$ are identical as multisets. We denote the number of elements in a multiset Γ by $|\Gamma|$. E.g., $|\{T, W, E, E\}| = 4$.

If φ maps formulas to formulas then, if $\Gamma = \{G_1, \ldots, G_n\}$, we write $\varphi\Gamma$ for the multiset $\{\varphi G_1, \ldots, \varphi G_n\}$. So $!\Gamma$ stands for $\{!G_1, \ldots, !G_n\}$, Γ^* for $\{G_1^*, \ldots, G_n^*\}$, et cetera.

We use 'rhs' and 'lhs' for respectively 'right-hand side' and 'left-hand side' (of the entailment sign).

The following conventions are used in distinguishing between the occurrences of formulas in a given logical rule, e.g., L $\rightarrow$:

$$\frac{\Gamma_1 \Longrightarrow \Delta_1, A \qquad B, \Gamma_2 \Longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \Longrightarrow \Delta_1, \Delta_2}.$$

The formula $A \rightarrow B$ is called the *main* formula of the rule with main connective $\rightarrow$; the occurrences A and B in the premises will be referred to as the *active* formulas; all other occurrences are said to be *passive* and will be referred to as the *context*. In the special case of a cut, active formula occurrences are also termed cutformulas.

In case of an identity axiom we say that *both* formula-occurrences are main.

We will restrict the use of the term 'main' to logical rules and identity axioms (we will sometimes speak of the *axiom rule*). In case of structural rules, if necessary, we speak of the *weakened*, respectively *contracted* formula.

Given an occurrence of a formula A in a sequent derivation π, we will speak of "the tree of A's ancestors in π", meaning the tree-like structure obtained by tracing upwards in π all formula occurrences corresponding to our initial A, up to the introductions of A by axiom, logical or weakening rule (these form the tree's leaves). The reader who wishes to do so, will easily provide the (long and boring) inductive definition.

In a rule

$$\frac{\vdots \quad \vdots}{\sigma' \quad \sigma''}\,\sigma \qquad \text{or} \qquad \frac{\vdots}{\sigma'}\,\sigma$$

σ is called the successor-sequent of σ', σ'' in π. Similarly we will speak of the successor in σ of a formula occurrence in σ', σ'' (formal definition again left to the reader; using the terminology of [3]: the formula occurrence A in σ is a successor of the formula occurrence A in σ', σ'' if and only if both occurrences are in the same *identity class*).

A *block* (of sequents) in π is a sequence of sequents $\sigma_0, \sigma_1, \ldots, \sigma_n$ in π such that σ_{i+1} is a successor of σ_i. Note that a block of sequents unambiguously defines a block of (successive occurrences of) *rules* in π.

Given an occurrence of a formula A in some sequent σ in a proof π, such that A has just been introduced (is *'newly born'*, i.e., is either main, or has just been weakened), we inductively define a sequence of occurrences $A_0, \ldots, A_n$ of A together with a block B_A of sequents $\sigma_0, \ldots, \sigma_n$, by

(1) $\sigma_0 := \sigma$, $A_0 := A$;
(2) if $\sigma_i \in \mathsf{B}_A$, A_i is not active, and σ_{i+1} is the successor of σ_i, then $\sigma_{i+1} \in \mathsf{B}_A$, and A_{i+1} is the successor of A_i in σ_{i+1};
(3) that's all.

Hence B_A will contain σ, σ's successor, et cetera, down to either π's concluding sequent, or a sequent in which A is *active*. We will refer to B_A as A's *main-active interspace* or *m-a-interspace*.

With each *active* occurrence of a formula A, we can associate a finite set $\{B_{A_1}, \ldots, B_{A_n}\}$ of *m-a*-interspaces, where each A_i corresponds to a leaf of A's tree of ancestors.

2.2. LK. Our sequent calculus **LK** (Appendix A) for classical logic is a variation on Gentzen's original system ([8]). As said, this is a highly general and symmetric calculus, with first and second order quantifiers, containing axioms for the constants and rules for all binary connectives both in 'additive' and in 'multiplicative' *style* (a terminology that stems from linear logic, as does our motivation to represent **LK** precisely in this manner).

As is well known, using the structural rules of weakening and contraction, these distinct formulations of the logical rules are *interderivable* (see also Section 5.7), and the two styles of the binary connectives are provably equivalent.[3] In fact, the

[3]It might amuse the reader to show (use cut!) that conversely these equivalences suffice to *derive* the structural rules (modulo a bit of weakening for the constants, as in linear logic, actually!).

'pre-linear' literature shows an even greater variety in formulations of the rules for the logical connectives, mostly without explicit motivation as to *why* the author(s) prefer(s) theirs over other possibilities. Often there simply *is* no reason to take this or that style, or, on closer reading, it turns out to be merely a matter of simplifying that or this technical detail.

However, it is a (simple) insight we owe to linear logic, that the additive and the multiplicative group of rules 'an sich' stand out, in the sense that they are *autonomous*: both are closed under the cut rule (the key-steps in the elimination procedure do not involve structural rules), allow expansion of non-atomic identity axioms (cf. the definition of η-proofs in 5.4) and, in the presence of negation, allow for 'De Morganisation' of the connectives. In order to respect this autonomy we ask derivations in **LK** to be 'homo-style', and therefore *distinguish* additive and multiplicative versions of all binary connectives, and of the constants. (Note that hetero-style fragments do not have the first two of the properties mentioned above.)

So, *classical (propositional) formulas* are defined in the usual inductive manner from an infinite set of atomic propositions $p_1, p_2, \ldots, p_n, \ldots$ and the constants $\top_m$, $\top_a$, $\bot_m$, $\bot_a$, the unary connective $\neg$ and the binary connectives $\overset{m}{\rightarrow}$, $\overset{a}{\rightarrow}$, $\vee_m$, $\vee_a$, $\wedge_m$, $\wedge_a$.

The definition of terms and the extension to first and second order formulas proceeds as usual.

In this paper we often deal with *coloured formulas*: each formula comes equipped with a mapping of its subformulas into a 'colour space' which we denote by $\{t, q\}$. When convenient, we will make explicit the colour ε of the formula itself by means of a superscript: A^{ε}.

When talking about formulas occurring in a sequent, without being explicit whether they occur on the rhs or lhs, they will be written with a subscript: A_s, with $s \in \{L, R\}$ (like *s*ide).

Thus also the meaning of A_s^{ε} will be clear.

We use $\bullet_\sigma$ to denote a(n unspecified) binary logical connective of unspecified style. I.e., $\bullet \in \{\rightarrow, \vee, \wedge\}$ and σ (like σtyle) $\in \{a, m\}$.

We use a 'bar' to indicate transposition within these two-element sets of symbols. For instance, $\bar{t} := q$, $\bar{R} := L$, $\bar{m} := a$.

2.3. LL. *Linear (propositional) formulas* are obtained from an infinite set of atomic propositions $p_1, p_2, \ldots, p_n, \ldots$ and the constants $\top$, **1**, $\bot$, **0**, using negation $(\cdot)^{\perp}$, the exponentials !, ?, and the binary connectives $\multimap$, $\rightsquigarrow$, $\wp$, $\oplus$, $\otimes$, &. The definition of terms and the extension to first and second order formulas again proceeds as usual.

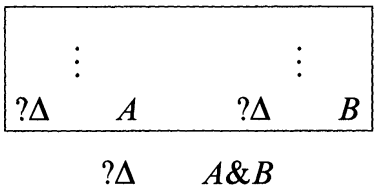
A formula in an **LL**-sequent $\Gamma \Longrightarrow \Delta$ is said to be of type '!' (respectively '?') if it is on the lhs (respectively rhs) and starts with a '?' or on the rhs (respectively lhs) and starts with a '!'. (Think of the sequent in its one-sided (only rhs) form, $\Longrightarrow \Gamma^{\perp}, \Delta$.)

A *modality* is a (possibly empty) sequence of exponentials. We denote modalities by μ, ν.

At some points in proofs we will use a bit of the stripping theory of [3], which is about how one can pluck exponentials in an **LL**-proof in such a way that the

'plucked' or 'stripped' proof is still a proof and reduces as the starting proof, up to dereliction and exponential commutation steps. Details in [3].

In Appendix B one finds the two-sided sequent-calculus **LL** for linear logic. We here mostly use this calculus as a convenient notation for proof nets. Actually we don't need the full apparatus of proof nets (as defined in [9]), but only the fragment consisting in: multiplicatives, exponentials (treated as in [2]), quantifiers (as in [11]) and so-called *tamed* additives, where binary additive boxes are asked to be of the form



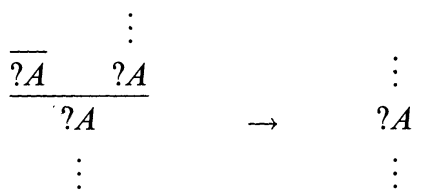
That is, all auxiliary doors are required to begin with a '?'. In the accordingly modified commutative reduction step the swallowed proof net is the !-box knocking at an auxiliary door. Thanks to the modified commutative reduction step this proof net system, denoted hereafter by taLL, is obviously stable by reduction.

THEOREM 1. *taLL normalization is noetherian and confluent.*

PROOF. Define the following mapping of **LL**-formulas to non-additive **LL**-formulas by

$$X^\circ = X, \qquad (A\&B)^\circ = !A^\circ \otimes !B^\circ, \qquad (A \oplus B)^\circ = ?A^\circ \parr ?B^\circ$$

and $^\circ$ leaves all non-additive symbols (negation, exponentials, multiplicative connectives and quantifiers) unchanged. (Exponentials with only weakening permission, as defined in [3], would be sufficient.) An easy computation then shows that $^\circ$ extends to a one-one embedding of taLL proof nets in non-additive **LL** proof nets, which is a morphism with respect to normalization. To be precise, one gets a morphism if non-additive proof nets are identified when they only differ up to so-called 'contracted weakenings':



Since the morphism is one-one and the normalization of non-additive proof nets is noetherian (cf. [9]) and confluent (cf. [2]) the theorem follows. ⊣

§3. Classical logic, what is wrong? In this section we will explain why naive cut-pushing steps are incompatible with a strongly normalizing and confluent cut-elimination procedure. For this there are two reasons, the *structural* and the *logical*. Adding suitable information to the proofs will allow the definition of a normalization scheme, which will be proven to be strongly normalizing and confluent in the next section.

3.1. Some information is missing. In **LK**, the naive cut eliminator will feel like Jean Buridan's donkey.[4]

3.1.1. *Structural dilemma.* Consider the problem of getting a cut-free proof of $A \vee_m A \Longrightarrow A \wedge_m A$, starting from:

$$\dfrac{\dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A \vee A \Longrightarrow A, A}}{A \vee A \Longrightarrow A} \qquad \dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A, A \Longrightarrow A \wedge A}}{A \Longrightarrow A \wedge A}}{A \vee A \Longrightarrow A \wedge A}$$

To begin with, clearly we have to *choose* whether the proof of the *left* premise of the cut or that of the *right* premise is going to be duplicated. Suppose we take the left option. We obtain:

$$\dfrac{\dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A \vee A \Longrightarrow A, A}}{A \vee A \Longrightarrow A} \qquad \dfrac{\dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A \vee A \Longrightarrow A, A}}{A \vee A \Longrightarrow A} \qquad \dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A, A \Longrightarrow A \wedge A}}{A, A \vee A \Longrightarrow A \wedge A}}{A \vee A, A \vee A \Longrightarrow A \wedge A}}{A \vee A \Longrightarrow A \wedge A}$$

Again, there are two possible choices (for each cut): either one persistently keeps on moving the same proof upwards and reaches the following normal form:

$$\dfrac{\dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A \vee A \Longrightarrow A, A}}{A \vee A \Longrightarrow A} \qquad \dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A \vee A \Longrightarrow A, A}}{A \vee A \Longrightarrow A}}{A \vee A, A \vee A \Longrightarrow A \wedge A}}{A \vee A \Longrightarrow A \wedge A}$$

or, all of a sudden, one decides to move the proofs of the right premises of the cuts. It's easy to see that random sequences of choices can result in this case in infinite reductions and even arbitrarily big normal forms. Hence the procedure has to be persistent (i.e., has to push a cut up to the leaves of the tree of ancestors of one of the cutformulas) in order to be noetherian.

Now the left-and-persistent option leads to a cut-free proof, which is quite different from the one given by the right-and-persistent option. Hence the procedure apparently has to know in advance which option to take (i.e., which subproof should be transported) in order to be confluent. The simplest way out of this left/right dilemma is to, brutely, fix a protocol beforehand, by *adding* the missing information to formulas. We call it the *colour*.

The resulting enriched system, the rules of which are just the good old rules of **LK** but now operating on coloured formulas,[5] will be referred to as $\mathbf{LK}^{tq}$. By convention colour t (respectively q) prescribes the left (respectively right) option (or protocol as we shall sometimes say). Accordingly an occurrence of a coloured

[4]Similar observations can be found in [10] and in [32].

[5]Alternatively, rules are supposed to preserve colours; yet in other words, colours may be assigned arbitrarily, provided they respect *identity classes*, cf. [3].

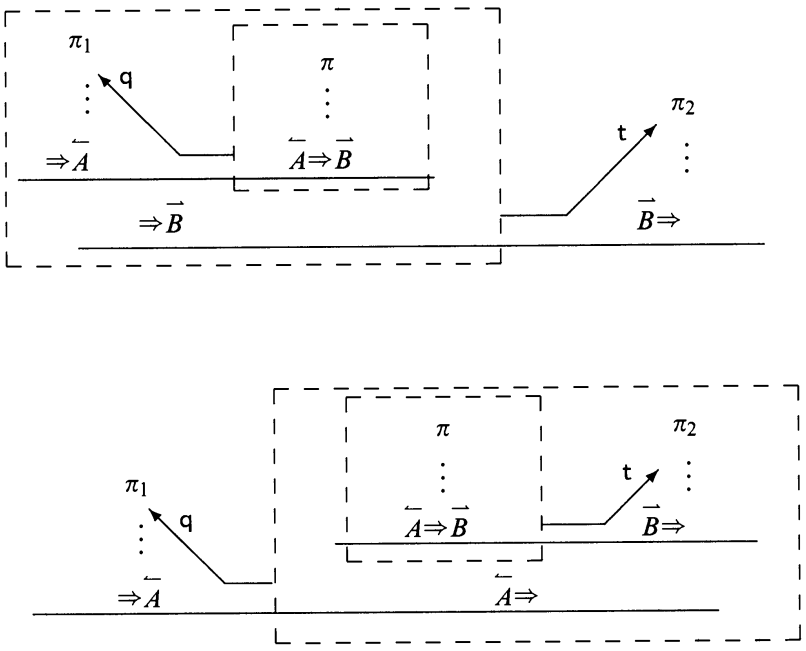


FIGURE 1. The q/t dilemma.

formula in the lhs (respectively rhs) of a sequent is said to be *attractive* if its colour is t (respectively q). This terminology is introduced in order to remind us that the proof of the sequent containing the non-attractive cutformula has to move first. We will often use an alternative iconic notation for A^t (respectively A^q), namely $\overrightarrow{A}$ (respectively $\overleftarrow{A}$).

Note that choosing colours has nothing to do with imposing a strategy. We do not select redexes, but rather the way we want to reduce them.

3.1.2. *Logical dilemma.* Consider two successive cuts on a given sequent $A_s, \ldots, B_{s'}$. The order in which the cuts are performed really matters in case neither of A_s and $B_{s'}$ is attractive, because as an easy computation shows, in that particular case, the proofs obtained by firing the two cuts will look seriously different depending on whether the cut on A_s is made above the one on $B_{s'}$ or not. For example, let us suppose that the sequent is $A^q, \ldots \Longrightarrow \ldots, B^t$. First cutting on A^q gives rise to the upper one of the two proofs in Figure 1, first cutting on B^t to the lower one.

Now in the first case π_1 only duplicates (or erases) π while π_2 duplicates (or erases) both π_1 and π, whereas in the second case π_1 duplicates (erases) both π and π_2, while π_2 only duplicates (erases) π.

In other words, the equivalence relation generated by these "seemingly innocuous permutations of rules"[6] will be incompatible with our protocol.

"Now why bother with that?", the reader may ask. "Just don't try to permute cuts, even if seemingly innocuous!" Right, but such concurrent cuts are the result of a multiplicative key-step, and to say that the order matters is just to say that we have to make a choice! For each multiplicative connective the dilemma will show up for *exactly* one distribution of the colours on the active formulas: (q, t) for $\overset{m}{\rightarrow}$,

[6]Quote from [10], p. 280.

(t, t) for $\vee_m$ and (q, q) for $\wedge_m$, i.e., in case neither of the formulas active in the unary rule is *attractive*.

Again, the simplest way out is to attach an additional bit of information: in what follows we assume the unary rules for the multiplicative connectives to come in two types, prescribing whether in the logical step the cut on A_s is made first or not. Alternatively, one might *orient* the multiplicative connectives, where the orientation fixes the order of the cuts in the key-steps. Unlike for the colour of a formula, however, it is not necessary for an *identity class* to respect orientations, which is the technical formulation of the fact that introducing oriented connectives is a solution to the logical dilemma, but more drastic than strictly necessary. The system obtained, having either oriented rules or connectives, will still be called $\mathbf{LK}^{tq}$.

3.2. The *tq*-protocol. In this subsection we define a normalization-scheme for $\mathbf{LK}^{tq}$, which suits the analysis of Section 3.1, and to which we will refer as the *tq-protocol*. In each instance of a cut rule in an $\mathbf{LK}^{tq}$-proof, the cutformula will be coloured either t or q. Thus, using our iconic notation, cuts are of one of the two following forms:

$$\frac{\vdots \qquad \vdots}{\Longrightarrow \vec{A} \qquad \vec{A} \Longrightarrow}{\Longrightarrow} \qquad \text{and} \qquad \frac{\vdots \qquad \vdots}{\Longrightarrow \overleftarrow{A} \qquad \overleftarrow{A} \Longrightarrow}{\Longrightarrow}.$$

Let us call the subderivation containing the *attr*active occurrence of the cutformula the *attr*acting subderivation.

DEFINITION 2 (*tq*-protocol). Reduction according to *tq*-protocol proceeds via two possible types of steps, 'structural' ones, S1 and S2, and 'logical' ones, L ('key-steps'):

– An L-step applies when both cutformulas are main in a logical rule, in which case we obtain as descendants one or two cuts on the immediate subformula(s) of the cutformula. The order in which these cuts are applied is determined by the orientation.

– In case no L-step is applicable, necessarily an S-step applies, which consists in 'transporting' one of the cut's subderivations up the tree of the cutformula's ancestors in the other one, duplicating it and contracting the context whenever passing an instance of contraction (or via the context of a binary additive rule); this process ends when reaching instances of introduction in an axiom, in which case the resulting 'axiom-cuts' are reduced *immediately*, when reaching an introduction by weakening, which are replaced by weakenings of the context formulas, or when reaching instances of introduction of the main connective of the cutformula.

Of course, now one needs to know *which* of the two subderivations has to move. This is decided by asking whether or not the attractive cutformula is main *in a logical rule*. If the answer is "yes!", we transport the attracting subderivation (S2); if it is "no!", we transport the other one (S1).

And that's it.

Let's define the *energy* of a cut c as the pair $(d(A), s(A))$ where $d(A)$ is the number of symbols in A and $s(A)$ is the type of the *tq*-reduction-step applicable to c: S1, S2

or L. By definition of those steps, when applied to c, the energy of (the descendants of) c decreases (energies are lexicographically ordered, with S1 > S2 > L). (Except, of course, in a second order quantifier L-step, where the possible increase of $d(A)$ is out of control.)

At this point the reader might try his left hand at proving the following.

THEOREM 3 (Weak normalization theorem). *In first order* $\mathbf{LK}^{tq}$ *the cut rule is redundant.*

In fact the induction argument can be kept surprisingly simple: suppose π is a proof and c a cut in π with cutformula A. Now, to deduce from the remark above the usual key-lemma in a cut-elimination proof 'à la Gentzen' (see, e.g., [41]), it suffices to show that the energy of (descendants of) any cut other than c in π never increases. Our forthcoming so-called 'stability-lemma' (Lemma 16) will establish this 'decrease of energy'-property.

Note that, in contrast to the usual proofs of cut-elimination, we do not need *explicitly* the notion of 'multi-cut'. In a certain sense that notion is *implicit* in our definition of the structural reduction steps.

This simple weak normalization proof is a foretaste of the 'good quality' of the tq-reduction scheme.

Note also that, following the tq-protocol, no "inconsistent" permutations of cuts (as described in the "logical dilemma") can occur. We leave it to the reader to verify that whenever one of both cutformulas is main, the reduction is independent of the cutfomula's colour ('$t = q$') and that adding to the tq-protocol an *axiom-step*, saying that a cut with an identity axiom may be removed whatever the colour of the cutformulas, is conservative over the above definition. (Also, the protocol is easily extended so as to include cuts on axioms for the constants. We will however not consider the case of the constants, neither here, nor in the sections to come and leave their treatment to the zealous reader.)

A remark as to why we ask the *immediate* reduction of axiom-cuts appearing in the course of an S-step: this in fact is a 'sine qua non' for strong normalization. Consider the following example:

$$\dfrac{\dfrac{A \Longrightarrow A \quad A \Longrightarrow A}{A \Longrightarrow A} \qquad A \Longrightarrow A}{A \Longrightarrow A}$$

where not absorbing the axiom gives rise to a cyclic reduction. Anyway, our strong normalization argument forces this absorption.

If we consider this procedure as defining a reduction of **LK** (i.e., we forget about the colours), then note that there is no ambiguity whatsoever in applying a structural step *except* for the choice, at each given instance of a cut, which of the two cutformulas is going to be the attractive one. Once the choice has been made, as in $\mathbf{LK}^{tq}$, the tq-protocol completely determines the structural phase of the reduction.

Having defined what we mean by 'normalizing' an $\mathbf{LK}^{tq}$-proof, let us reconsider the logical dilemma. Recall that there is, possibly, a dilemma, only in case *both* formulas are attractive: then it *might* happen that the two cuts do 'not commute'. We are now in a position to verify this in detail, by computation. Let's have another

look at a problematic situation, e.g., as in Figure 1. If we denote the two possible configurations by Π_1 and Π_2, and write Π_i^A, Π_i^B (respectively Π_i^{AB}) for the reduct of Π_i, obtained by performing the appropriate *tq*-reduction-step for the cut on A, B (respectively for the cut on A *and* for the cut on B), then one can show by inspection of all possible cases that the following holds.

LEMMA 4. *If A in π_1 (or B in π_2) is main or non-attractive, then $\Pi_1^A = \Pi_2^A$ (or $\Pi_1^B = \Pi_2^B$); hence if at least one of A in π_1, B in π_2 is non-attractive or main, then $\Pi_1^{AB} = \Pi_2^{AB}$.*

However, when the conditions of the lemma are not fulfilled, equating Π_1 and Π_2 is inconsistent with our protocol. For instance, take π_1 and π_2 to be respectively:

$$\dfrac{\begin{array}{c}\pi_1' \\ \vdots \\ \Gamma\end{array}}{\Gamma, A} \qquad \dfrac{\begin{array}{c}\pi_2' \\ \vdots \\ \Delta\end{array}}{\Delta, B}$$

with A and B both attractive, then one obtains for Π_1^{AB} and Π_2^{AB} respectively:

$$\dfrac{\begin{array}{c}\pi_2' \\ \vdots \\ \Delta\end{array}}{\Delta, \Gamma'} \qquad \dfrac{\begin{array}{c}\pi_1' \\ \vdots \\ \Gamma\end{array}}{\Delta', \Gamma}$$

where Γ' (respectively Δ') is the union of contexts in the conclusion of π_1' and π (respectively π_1' and π). As π_1' and π_2' are arbitrary, equating Π_1 and Π_2 would amount to identifying all proofs of a same sequent, a total collapse of the calculus.

§4. Plethoric embedding.

4.1. Linear logic: djs' boa deconstructor. The key to the use of linear logic as a tool for the study of derivations in classical logic is the following simple *property of forgetfulness*: if, given some derivation π in **LL**, we replace all linear connectives by their classical analogues, i.e., $\multimap$, $\rightsquigarrow$, $\otimes$, $\&$, $\wp$, $\oplus$ respectively become $\overset{m}{\to}$, $\overset{a}{\to}$, $\wedge_m$, $\wedge_a$, $\vee_m$, $\vee_a$, erase all the exponentials and possible resulting repetitions of sequents, then what we get is a derivation in **LK**. We will speak of the (classical) *skeleton* $\mathrm{sk}(\pi)$ of π.

More so, there is a converse to this property:

THEOREM 5 (decoration theorem). *Each* **LK***-derivation is the classical skeleton of an* **LL***-derivation.*

An **LL**-derivation is said to be a *decoration* of its classical skeleton. Decorations are far from unique, and in fact, given an **LK**-derivation, the collection of its decorations is infinite.

In order to prove that the *tq*-normalization defined in the previous section is noetherian and confluent, we will exhibit a decoration of $\mathbf{LK}^{tq}$ such that the *tq*-normalization of a proof can be *simulated* by the normalization of (the proof net associated to) its decoration.

LL seems here relegated to the ancillary task of proving these properties of our *tq*-protocol. But of course things went the other way round: we were looking for decorations of **LK**, *because*, by the property of forgetfulness, the 'pull-back'[7] of **LL**-normalization through *any* decoration would be a normalization scheme for **LK**. This was how we found the *tq*-protocol.

This indicates also the *limits* of the method: for instance, relaxed *tq*-reduction, where colours of the cut formula are randomly chosen at each structural step, probably is terminating (because it seems to be the sequent calculus counterpart of Barbanera and Berardi's symmetric reduction in λ-calculus, proven to be terminating in [1]) but the decoration method cannot be applied directly; also, reduction based on Gentzen's cross-cuts seems to be out of reach, due to the 'heavy asymmetry' imposed by the **LL**-way of treating non-linear cuts (cf. [15] for an overview of ways of reducing classical proofs).

So let's look for decorations of **LK**!

To get a normalization scheme that is definable in a reasonable way, it seems good politics to restrict our attention to decorations that can be obtained in a *uniform* way, namely by applying *inductively* a modal translation $(\cdot)^*$ to **LK**-derivations: we replace each classical formula A by a linear formula A^*, obtained from A by replacing classical connectives by their linear analogues, and prefixing each subformula of A by a modality; a sequent $G_1, \ldots, G_n \Longrightarrow D_1, \ldots, D_m$ becomes $\mu_1 G_1^*, \ldots, \mu_n G_n^* \Longrightarrow \nu_1 D_1^*, \ldots, \nu_m D_m^*$ (we call the μ_i, ν_j, the *modal prefixes*, or *context modalities*). Inductively means, at the level of formulas, that the translation of A is defined uniformly in terms of the translation of A's subformulas and does not depend on the context in which A occurs; at the level of proofs, that it is determined by no other than merely *local* information, namely the instance of the **LK**-rule to be translated into a correct instance of the corresponding **LL**-rule: the translation of proofs should be 'history-free'.

The purpose of the game is to obtain in the end an **LL**-derivation whose classical skeleton is the **LK**-derivation that we started from. Therefore, the only degree of freedom we have in translating a given instance of an **LK**-rule is the application of some number of exponential dereliction- and promotion-rules before and/or after the instance of the corresponding **LL**-rule: we will speak of *pre-* and *post-modalizations*.

The *structural rules* in **LK** oblige us to guarantee that all formulas occurring are of type '?' (i.e., begin with a '!' if on the lhs, or a '?' if on the rhs); the *cut rule*, that the modal prefixes μ_R (for formulas at the rhs) and μ_L (for formulas at the lhs), inductively given for the occurrences of the cutformula in the premises, can be *unified*, i.e., either μ_R is a suffix of μ_L or μ_L is a suffix of μ_R. A pair of modalities satisfying these conditions is said to be *adequate*.

There are *two* minimal ways to inductively decorate an occurrence of a cut in an **LK**-derivation, one corresponding to the adequate pair $\mu_R^t = ?$, $\mu_L^t = !?$, the other to its dual $\mu_R^q = ?!$, $\mu_L^q = !$. These two decorations correspond to the two possible choices in the 'structural dilemma'. In either case, in the corresponding proofnet

[7]Suppose π is an **LK**-proof and $d(\pi)$ is a decoration of π. Then each sequence of reducts of $d(\pi)$ defines a possible sequence of reducts of π by the property of forgetfulness. Thus, each decoration defines a (finite) subset of the possible reductions of its skeleton (whatever a reduction in **LK** may be).

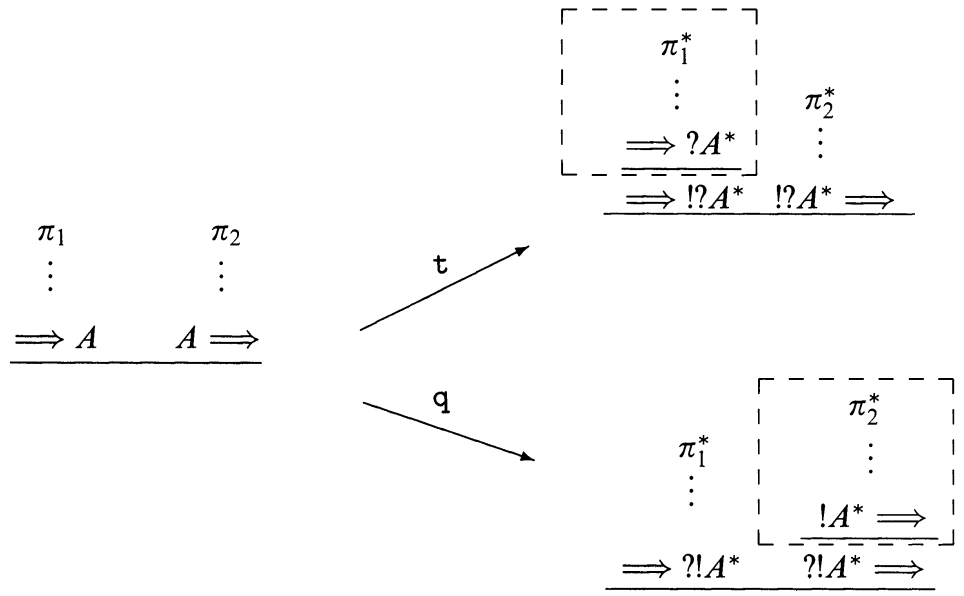


FIGURE 2. Cut decoration.

exactly *one* of the premises is a *box* (cf. Figure 2) which in the decoration corresponds to the attracted derivation in $\mathbf{LK}^{tq}$. Note that the decoration also shows that colours have to respect identity classes: it is impossible to 'unify' $(!)?A$ and $(?)!A$. All other adequate pairs are shown, in [40], to define by pull-back just the same tq-protocol.

4.2. Decorating $\mathbf{LK}^{tq}$. We will only consider *implication, negation* and *universal quantification*, which, as we'll see below, we can do without loss of generality.

DEFINITION 6 (plethoric translation). We define a mapping D of coloured classical formulas to linear formulas: $D(X^t) = D(X^q) = X$ for atoms, and

$$D(A^\varepsilon \overset{m}{\to} B^{\varepsilon'}) = \mu^\varepsilon D(A) \multimap \mu^{\varepsilon'} D(B) \qquad D(A^\varepsilon \overset{a}{\to} B^{\varepsilon'}) = \mu^\varepsilon D(A) \rightsquigarrow \mu^{\varepsilon'} D(B),$$

except when $(\varepsilon, \varepsilon') = (q, t)$ (i.e., when the logical dilemma occurs) in which case:

$$D(A^q \overset{m}{\to} B^t) = \begin{cases} \mu^q D(A) \multimap ?\mu^t D(B) \\ !\mu^q D(A) \multimap \mu^t D(B) \end{cases}$$

depending on the orientation.

$$D(\forall X\, A^\varepsilon) = \forall X\, \mu^\varepsilon D(A) \qquad D(\neg A^\varepsilon) = (\mu^\varepsilon D(A))^\perp$$

where μ^ε is the longest of μ^ε_L and μ^ε_R (i.e., $\mu^t = !?$ and $\mu^q = ?!$).

Explicitly, D is defined by:

$A \overset{m}{\to} B$	t	q
t	$!?A \multimap !?B$	$!?A \multimap ?!B$
q	$?!A \multimap ?!?B$ $!?!A \multimap !?B$	$?!A \multimap ?!B$

$A \overset{a}{\to} B$	t	q
t	$!?A \rightsquigarrow !?B$	$!?A \rightsquigarrow ?!B$
q	$?!A \rightsquigarrow !?B$	$?!A \rightsquigarrow ?!B$

$\forall XA$	
t	$\forall X!?A$
q	$\forall X?!A$

$\neg A$	
t	$(!?A)^\perp$
q	$(?!A)^\perp$

Consider the two translations in the (q,t) case of $\overset{m}{\to}$: the first one, $?!A \multimap ?!?B$, corresponds to the orientation "A-cut under the B-cut" during the implication logical step; the second one, $!?!A \multimap !?B$, to the opposite orientation. If one decides to use oriented connectives, the two translations correspond to these two connectives. Observe, though, that the two translations can be 'unified' as $!?!A \multimap ?!?B$. This *proves* that the orientation need not be preserved by identity classes; hence that it is sufficient to distinguish two versions of the unary rules for the multiplicative connectives, which will correspond to the *two* ways in which an instance of such a rule can be decorated:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots,!A \Longrightarrow ?B,\dots}{\dots,?!A \Longrightarrow ?B,\dots}}{\dots,!?!A \Longrightarrow ?B,\dots}}{\dots,!?!A \Longrightarrow !?B,\dots}}{\dots,!?!A \Longrightarrow ?!?B,\dots} \qquad \text{or} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dots,!A \Longrightarrow ?B,\dots}{\dots,!A \Longrightarrow !?B,\dots}}{\dots,!A \Longrightarrow ?!?B,\dots}}{\dots,?!A \Longrightarrow ?!?B,\dots}}{\dots,!?!A \Longrightarrow ?!?B,\dots}$$

So one sees how the dilemmas described in Section 3 become obligations to *choose* when inductively decorating an **LK**-derivation: the 'structural dilemma' is equivalent to either t- or q-decorating the (identity class of the) corresponding cutformulas, the 'logical dilemma' amounts to a non-determinism in the translation of formulas that *might* get involved in a 'problematic' cut, and choosing one or the other corresponds to choosing an orientation of the original rule, or connective, if one prefers.

Also, note that, given a linear formula of the form $D(F)$, one unambiguously deduces the colouring of F's subformulas; indeed, a subformula G of F is t if and only if in $D(F)$ it is immediately prefixed by a '?', it is q if and only if it is immediately prefixed by '!'.

THEOREM 7 (plethoric embedding). *There is an inductive decoration that maps proofs of* $\mathbf{LK}^{tq}$ *to proofs of* **LL**. *A proof* π *of* $\Gamma^t, \Lambda^q \Longrightarrow \Delta^t, \Sigma^q$ *is mapped to a proof* $D(\pi)$ *of* $!?D(\Gamma), !D(\Lambda) \Longrightarrow ?D(\Delta), ?!D(\Sigma)$ *the skeleton of which is equal to* π.

PROOF. Cut, identity axiom and structural rules are unproblematic, by adequacy of the chosen context modalities. As to the logical rules, we don't want a case-by-case inspection, so let's put it abstractly. First consider the multiplicative case which is the hardest.

We want to prefix the active formulas in the premises of the unary and binary multiplicative rule of a given connective so that we get the same final modality for the formula on both sides. In other words, we have to address the following 'unification'-problem: find modalities $(\Box_A, \Box_B, \mathsf{O}_A, \mathsf{O}_B)$ such that: (1) $\Box_A \mu_s^\varepsilon = \mathsf{O}_A \mu_{\bar{s}}^\varepsilon$ and $\Box_B \mu_{s'}^{\varepsilon'} = \mathsf{O}_B \mu_{\bar{s}'}^{\varepsilon'}$; (2) the following are derivable:

$$\dfrac{\mu_s^\varepsilon A_s, \mu_{s'}^{\varepsilon'} B_{s'}}{\Box_A \mu_s^\varepsilon A_s, \Box_B \mu_{s'}^{\varepsilon'} B_{s'}} \qquad \dfrac{\mu_{\bar{s}}^\varepsilon A_{\bar{s}}}{\mathsf{O}_A \mu_{\bar{s}}^\varepsilon A_{\bar{s}}} \qquad \dfrac{\mu_{\bar{s}'}^{\varepsilon'} B_{\bar{s}'}}{\mathsf{O}_B \mu_{\bar{s}'}^{\varepsilon'} B_{\bar{s}'}}.$$

Solutions to (1) always exist because of the suffix property we asked for both adequate pairs (a property which is btw obviously necessary); rules in (2) are correct if and only if at most one of $(\Box_A, \Box_B)$ is of type ! (because the context is

of type ?). In case $s = L$, $s' = R$ (in other words, in case the connective under consideration is $\xrightarrow{m}$) the shortest prefixes satisfying (1) and (2) are given by the following table:

$(\Box_A, \Box_B)$	t	q
t	$(.,!)$	$(.,.)$
q	$(?,?!)$ $(!?,!)$	$(?,.)$

where . means the void modality. In the critical (q,t) case there are two minimal solutions; see below.

In the additive case the condition (2) is always fullfilled for there is only one formula active in the unary rule, so the shortest prefixes' table is just the trivial unification of adequate pairs:

$(\Box_A, \Box_B)$	t	q
t	$(.,!)$	$(.,.)$
q	$(?,!)$	$(?,.)$

These two tables define exactly the pletho-translation of both styles of implication. The cases of quantifiers and negation are obvious. ⊣

In fact, one sees that the translation is entirely determined by asking for an inductive decoration of $\mathbf{LK}^{tq}$-derivations (i) which uses as little exponentials as possible and (ii) in which ε-coloured formulas are prefixed by μ_s^ε. We speak of a *plethoric* translation, because nevertheless the number of exponentials occurring is considerable.

In order to treat the complete propositional fragment, one can use the following "duality-correspondence" between pletho translations of the connectives:

$$\varepsilon \quad \boxed{\mu A \left\{ \begin{matrix} \multimap \\ \rightsquigarrow \end{matrix} \right\}^{\varepsilon'} \nu B} \qquad \bar{\varepsilon} \quad \boxed{\mu^\perp A \left\{ \begin{matrix} \wp \\ \oplus \end{matrix} \right\}^{\varepsilon'} \nu B} \qquad \varepsilon \quad \boxed{\mu A \left\{ \begin{matrix} \otimes \\ \& \end{matrix} \right\}^{\bar{\varepsilon}'} \nu^\perp B}$$

The dual $\mu^\perp$ of a modality μ is obtained by replacing all ! by ?, all ? by !; e.g., to find the inductive clause for $A^q \vee_m B^t$, it suffices to take the modalities in the clause for $A^t \xrightarrow{m} B^t$ (being !?, !?), and dualize the first of these, thus finding $?!A \wp !?B$. Similarly, it suffices to consider only one of the quantifiers.

In order to obtain a *converse* to Theorem 7, we have to ask whether, given an **LL**-derivation π of a sequent $!?D(\Gamma), !D(\Lambda) \Longrightarrow ?D(\Delta), ?!D(\Sigma)$, it is possible to assign colours to all formulas occurring in $\mathsf{sk}(\pi)$ in such a way that the resulting derivation is an $\mathbf{LK}^{tq}$-derivation of $\Gamma^t, \Lambda^q \Longrightarrow \Delta^t, \Sigma^q$. Note that in general this will be false: $!A \Longrightarrow ?A$ is clearly **LL**-derivable, but $A^q \Longrightarrow A^t$ is not $\mathbf{LK}^{tq}$-provable, for atomic A. However, this is the *only* problematic case, which is easily fixed by restricting the use of identity axioms to modalized formulas. We then can prove the following, the detailed verification of which is left to the reader.

PROPOSITION 8. *A sequent $\Gamma^t, \Lambda^q \Longrightarrow \Delta^t, \Sigma^q$ is derivable in $\mathbf{LK}^{tq}$ if and only if $!?D(\Gamma), !D(\Lambda) \Longrightarrow ?D(\Delta), ?!D(\Sigma)$ is derivable in* **LL** *with identity axioms restricted to modalized formulas only.*

4.3. Strong normalization. Let π be a proof in $\mathbf{LK}^{tq}$ and $D(\pi)$ denote its pletho-decoration as in Theorem 7. Let c be a cut in π with cutformula A^t, say. Then its decoration is as in:

$$\dfrac{\dfrac{\vdots}{\Longrightarrow ?D(A)}}{\Longrightarrow !?D(A)} \qquad \dfrac{\vdots}{!?D(A) \Longrightarrow}$$
$$\Longrightarrow .$$

Suppose now S2 applies to c in π, then the attractive cutformula is main in a logical rule, hence in this case the decoration of c is:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Longrightarrow ?D(A)}}{\Longrightarrow !?D(A)} \qquad \dfrac{\dfrac{\vdots}{D(A) \Longrightarrow}}{?D(A) \Longrightarrow}}{!?D(A) \Longrightarrow}$$
$$\Longrightarrow .$$

Therefore one can perform a dereliction reduction-step in **LL** to get the "adjusted" decoration:

$$\dfrac{\dfrac{\vdots}{\Longrightarrow ?D(A)} \qquad \dfrac{\vdots}{D(A) \Longrightarrow}}{?D(A) \Longrightarrow}$$
$$\Longrightarrow .$$

Next, suppose L applies to c. Then both cutformulas are main in a logical rule, and therefore the decoration of c is as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Longrightarrow D(A)}}{\Longrightarrow ?D(A)} \qquad \dfrac{\dfrac{\vdots}{D(A) \Longrightarrow}}{?D(A) \Longrightarrow}}{!?D(A) \Longrightarrow}$$
$$\Longrightarrow .$$

Now one can perform *two* consecutive dereliction reduction-steps in **LL** to get the adjusted decoration:

$$\dfrac{\vdots}{\Longrightarrow D(A)} \qquad \dfrac{\vdots}{D(A) \Longrightarrow}$$
$$\Longrightarrow .$$

Let the *adjusted* pletho-decoration of π, denoted by $D^-(\pi)$ be the taLL proof net associated to the sequent calculus **LL**-proof where decoration of cuts has been adjusted as above, i.e., appropriate dereliction reduction-steps have been performed for each cut to which S2 or L applies. The effect of that adjustment is that the

cutformula in the adjusted decoration has two (respectively one, zero) 'surface' exponentials exactly in case S1 (respectively S2, L) applies.

THEOREM 9 (simulation theorem). *The adjusted pletho-decoration mapping proofs of* $\mathbf{LK}^{tq}$ *to proof nets is a homomorphism with respect to normalization. That is, for any proof* π *of* $\mathbf{LK}^{tq}$ *and any reduction of* π *to* π', *there exists a reduction from* $D^-(\pi)$ *to* $D^-(\pi')$. *This reduction is non void except possibly in the case of a cut on* $\neg A$ *if an L-step applies.*

PROOF. The proof is only a matter of simulating each type of reduction-step, which is verified by careful computation.

As to the nature of that image reduction: note again that either the energy of (the descendants of) a given cut decreases when it is reduced, and since the decoration is adjusted, also the number of exponentials of (the descendants of) cutformulas in the decoration decreases, or the main connective disappears (in case the reduction-step is L). In both cases there is a change in the corresponding proof net except in the case mentioned above exactly when the descendant of the cut on $\neg A$ has maximal energy. ⊣

The following then is a consequence of our taLL normalization theorem in Section 2 and the simulation theorem above, considering that invisible steps obviously cannot accumulate forever.

THEOREM 10 (strong normalization). *The tq-protocol of* 3.2 *is strongly normalizing.*

Via our decorations we can use Girard's coherent semantics for linear logic ([9]) as a denotational semantics for $\mathbf{LK}^{tq}$.

PROPOSITION 11 (pletho or D-semantics). *Let* π *be a proof of* $\mathbf{LK}^{tq}$, *and let* $D(\pi)^*$ *be the denotation of* $D(\pi)$ *in linear logic's coherent semantics. This interpretation is non-trivial and invariant under tq-normalization (in other words,* $\star \circ D$ *is a denotational semantics for* $\mathbf{LK}^{tq}$*).*

PROOF. First note that since the proof net associated to $D(\pi)$ reduces to $D^-(\pi)$ they have the same semantics. Now by the simulation theorem, it is sufficient to show that the coherent semantics of a taLL proof net is invariant under any elementary reduction step. As taLL reduction steps differ from the usual ones only in the commutative reduction step for tamed additives, we need only check that this precise step leaves the semantics unchanged. And this evidently is the case. ⊣

4.4. Church-Rosser express. Normal forms of a given proof are mapped by the pletho embedding to the same proof net, but a same proof net may have several sequentializations. So this is not enough to get unicity of normal forms. The trick is to overload the modalization used in the pletho embedding to the effect that the resulting proof nets are 'rigid', i.e., have a(n almost) unique sequentialization.

So define another embedding using 'fatter' adequate pairs $\mu_L^{t'} = !?!$, $\mu_R^{t'} = ?!$ and $\mu_L^{q'} = !?$, $\mu_R^{q'} = ?!?$. That it defines a decoration comes from the fact that $\mu_s^{t'} = \mu_s^t!$ and $\mu_s^{q'} = \mu_s^{q'}?$, hence the unification problem now to solve will only be a trivial 'shifting' of the preceding one and will admit the shifted solutions of the pletho

one as particular solutions. The simulation theorem will still hold for just the same reason (or because the added exponentials are superfluous as proved in [40]). Now observe that each main formula in a logical rule will thus get a promotion in the post-modalization phase. From the confluence of proof nets with tamed additives then follows:

THEOREM 12 (confluence). *Two tq-normal forms of an* $\mathbf{LK}^{tq}$*-proof are equal up to permutations of structural rules.*

A direct proof of local confluence would do as well, but would be tedious to provide.

4.5. Weak equivalence. Now that we have installed a noetherian and confluent normalization, let us define a quotient of the syntax that is compatible with that normalization and which will greatly optimize our syntactic grill-time ([14]).

Two proofs are said to be *weakly equivalent* if they differ only up to contracted weakenings, re-arrangement of multiple contractions on the same formula (associativity, commutativity) and permutations of structural rules as, e.g., for a contraction and a binary multiplicative rule:

$$\dfrac{\dfrac{\ldots, A^s, A^s, \ldots}{\ldots, A^s, \ldots} \quad \ldots}{\ldots, A^s, \ldots} \overset{w}{\approx} \dfrac{\dfrac{\ldots, A^s, A^s, \ldots \quad \ldots}{\ldots, A^s, A^s, \ldots}}{\ldots, A^s, \ldots}$$

and for a contraction and a binary additive rule:

$$\dfrac{\dfrac{\ldots, A^s, A^s, \ldots}{\ldots, A^s, \ldots} \quad \dfrac{\ldots, A^s, A^s, \ldots}{\ldots, A^s, \ldots}}{\ldots, A^s, \ldots} \overset{w}{\approx} \dfrac{\ldots, A^s, A^s, \ldots \quad \ldots, A^s, A^s, \ldots}{\dfrac{\ldots, A^s, A^s, \ldots}{\ldots, A^s, \ldots}}$$

We will write $R \overset{n}{\approx} S$ to indicate that the taLL proof nets R and S differ only up to box/contraction or box/weakening permutations, contracted weakenings, and re-arrangement of contractions (associativity or commutativity).

By the *exponential development* of an exponential cut we mean the reduction of the cut and its residues, in one go, up to the leaves of the tree of the cutformula's ancestors, where one then performs the appropriate steps: axiom, box/dereliction steps or erasing.

If $R \overset{n}{\approx} S$ and R reduces to R' by such an exponential development then the same step applies to S, yielding a reduct S' that is $\overset{n}{\approx}$-equivalent to R' (note that additive box/structural permutations involve contracted weakenings). This is of course also the case for logical and axiom steps. Hence, because one can always reach the normal form with exponential developments as only exponential steps:

LEMMA 13. *If* R *and* S *are* $\overset{n}{\approx}$*-equivalent then so are their normal forms.*

In fact the syntax of proof nets may be reworked so that members of a same $\overset{n}{\approx}$-class are identified syntactically; this "new syntax" was considered in [39, 7].

PROPOSITION 14 (weak equivalence is *tq*-stable). *If two* $\mathbf{LK}^{tq}$*-proofs are weakly equivalent, then so are their normal forms.*

PROOF. Take once more the rigid embedding used in the previous subsection. Let us call it R (like rigid). Observe that $\pi \overset{w}{\approx} \pi'$ if and only if $R(\pi) \overset{n}{\approx} R(\pi')$ (the "only if" part is true inasmuch as *decorations* are concerned, only the "if" part needs rigidity).

So if $\pi \overset{w}{\approx} \pi'$, then $R(\pi) \overset{n}{\approx} R(\pi')$, hence $\mathrm{nf}(R(\pi)) \overset{n}{\approx} \mathrm{nf}(R(\pi'))$. Then, by the (adaptation to the rigid case of the) simulation theorem $R(\pi)$ and $R(\pi')$ reduce to the proof nets $R(\mathrm{nf}(\pi))$ and $R(\mathrm{nf}(\pi'))$. Those being cut-free, by unicity of normal forms for proof nets, are nothing but $\mathrm{nf}(R(\pi))$ and $\mathrm{nf}(R(\pi'))$ which are by the lemma above $\overset{n}{\approx}$-equivalent. Therefore $R(\mathrm{nf}(\pi)) \overset{n}{\approx} R(\mathrm{nf}(\pi'))$. Whence finally, by the first observation, $\mathrm{nf}(\pi) \overset{w}{\approx} \mathrm{nf}(\pi')$. ⊣

This last proposition has many an enjoyable consequence. It implies e.g., that *tq*-normalization will 'work' for a calculus with implicit formulation of the structural rules.

Also, when eliminating a cut in the following situation

$$\dfrac{\begin{array}{cc} & \pi' \\ \pi & \vdots \\ \vdots & A, A, \Gamma' \Longrightarrow \Delta' \\ \Gamma \Longrightarrow \Delta, A & \overline{A, \Gamma' \Longrightarrow \Delta'} \end{array}}{\Gamma, \Gamma' \Longrightarrow \Delta, \Delta'}$$

in order to make it commute with the contraction, one has to duplicate π and contract all formulas in Γ and Δ. But in which order? The proposition above shows that this order is irrelevant. The same remark applies in case we commute a cut with a weakening on the cutformula, and replace it by a number of weakenings on context-formulas. In fact those boo-boo's don't even find their reflection in the pletho decorations.

Last, no finer Church-Rosser property than the one up to permutation of structural is interesting since those permutations are included in $\overset{w}{\approx}$-equivalence.

This 'weak equivalence' is strictly weaker than the equivalence induced by the coherent semantics (whence its name):

PROPOSITION 15. *If two* $\mathbf{LK}^{tq}$*-proofs are weakly equivalent, then their D-semantics are equal.* (*The converse does not hold.*)

PROOF. Note that, when $\pi \overset{w}{\approx} \pi'$, then clearly $D(\pi) \overset{n}{\approx} D(\pi')$. On the other hand, for any two proof nets, $R_1 \overset{n}{\approx} R_2$ implies that their coherent semantics are equal. The converse is not true, as pletho-decorations of proofs that differ in the order of two consecutive logical rules might have the same associated proof net. Take for example two proofs that differ only in the order of two consecutive right multiplicative t-implication introductions, the active formulas of which are coloured in the 'anti-critical' way (i.e., $A^t \Longrightarrow B^q$). ⊣

Hereafter, whenever convenient, we will consider $\mathbf{LK}^{tq}$-proofs up to weak equivalence.

§5. Stringent embedding.

5.1. So far. So far we have made an inventory of the precise points at which cut elimination in an **LK**-proof π is ambiguous (i.e., forces the eliminator to choose), and saw that each of the resulting 'dilemmas' corresponds on the nose to a *different* faithful interpretation of π as a proof in linear logic. By *adding beforehand* the information that possibly might be needed in the course of the normalization we defined a completely *unambiguous* procedure of cut elimination: when applying the tq-protocol to an $\mathbf{LK}^{tq}$-derivation, for each occurrence of a redex there is one and only one reduction-step that applies. Note also that each $\mathbf{LK}^{tq}$-proof is mapped to its computational twin in **LL** by the pletho embedding.

However, we *did* quite drastically change the notion of formula: the colours and orientations we added do not correspond to *anything* related to the classical *truth* of a given formula. As is the distinction of *two* versions of each unary multiplicative rule, these are purely formal, proof theoretical artifacts, enabling us to get a grip on the normalization of classical proofs. Note that we might opt for *uniform* colouring (all t, or all q), and for a *uniform* treatment of the problematic logical cuts (always 'A-cut above B-cut', or always 'B-cut above A-cut'), in which case colouring and orientation become obsolete, and we simply have *different* procedures for normalizing **LK**-derivations, each of which is strongly normalizing!

In fact, also the distinction 'additive/multiplicative' might be considered a proof theoretical artifact, and as such it is clearly related to the 'orientation'-information. Indeed, for *additive* versions of the binary connectives there *is* no logical dilemma: an additive connective has a natural orientation when engaged in a cut, as only *one* of its immediate subformulas will be present! Or, alternatively, one can consider the distinction of two versions of the unary multiplicative rules as adding an additive flavour to the multiplicative connective, which in its additive guise naturally demands two unary rules.

When using additive connectives, one imposes a restriction on the form of **LK**-derivations, which implicitly solves the logical dilemma. Contrariwise, there doesn't seem to exist a restriction on the form of **LK**-derivations that implicitly solves the *structural* dilemma. Brute force restrictions like 'at most one formula at the lhs' result in systems that are not complete for classical provability. Yes, the need to *add* information in order to control the structural phase of normalization appears the minimum price due for classical logic's deep symmetries: colouring is present in **LL** in the distinction between '!' and '?'. Another way to say this (cf. [10]) is that proof theory recently uncovered a hitherto hidden intrinsic property of classical formulas: they all come in pairs A^t, A^q. In *proofs* A^ε can interact with A^ε only: a classical formula A *has* to reveal itself as being either q or t during normalization. On the level of provability, however, there is no distinction: A^q is provable if and only if A^t is.

Hence we'll have to stick to $\mathbf{LK}^{tq}$, at least colour-wise.

But what about the distinction of two types of multiplicative unary rules, or the orientation of multiplicative connectives? As we saw above, there are many possible ways out, and, until proven otherwise, every one of them may claim to be *the* solution. So it should no longer come as a surprise to the reader that at this point there appear that many candidates for a computationally sound classical logic.

5.2. So what? Apart from uniform orientation, taking all connectives additive, or taking additive connectives only in the critical cases, there is a more fertile way to avoid the 'logical dilemma'. It starts from the observation made in Lemma 4: *whenever at least one of the two formulas in the problematic two-premise rule is main, there is no logical dilemma.* We are going to show that it is possible to restrict $\mathbf{LK}^{tq}$-derivations in such a way that this condition *always* is fulfilled, and moreover that nothing is lost.

As in setting up the *tq*-protocol, we were guided by the existence of an embedding into linear logic, an embedding that *realizes* the above restrictions. We call the system obtained by 'pull-back' $\mathbf{LK}^{\eta}$. Again the following presentation will violate the historical order. After some necessary technical preliminaries, we will define $\mathbf{LK}^{\eta}$ *inside* $\mathbf{LK}^{tq}$ (Section 5.5). The construction 'by pullback' via the corresponding embedding into **LL** is described in Section 5.6.

The restriction defining $\mathbf{LK}^{\eta}$, when made to bear upon proofs of $\mathbf{LJ}^{t}$ (the intuitionistic t-sequent calculus, i.e., at most one formula at the rhs and all formulas coloured t), results in a system that is very close to natural deduction, a system that was referred to as **ILU** in, e.g., [5, 4], or **LJT** in [15]. It is not a bad intuition to think of $\mathbf{LK}^{\eta}$ as a classical natural deduction.

5.3. Stability. In the following we will investigate the effect of *tq*-reductions on main-active interspaces (cf. Section 2.1) having a specific, simple, structure, namely having the form $\mathsf{B}_A = \sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n$, where all $\sigma_i \neq \sigma_0$ are structural rules on formulas other than A_i, and σ_n's successor a is either a *logical rule* or a *cut*:

$$\begin{array}{l} \vdots \\ \hline \\ \sigma_0 : \Gamma, A_s^{\varepsilon} \quad \mathtt{m/w} \\ \hline \\ \vdots \\ \hline \\ \sigma_n : \Gamma', A_s^{\varepsilon} \quad \mathtt{s} \\ \hline \\ \vdots \end{array}$$

We will refer to blocks of this kind as *lucid m-a*-interspaces.

If $n = 0$ and moreover A is main, we say that the (lucid) block is *flat*; if all $\sigma_i \neq \sigma_0$ are weakenings, we will call the block *w-lucid.*

Let π' denote a one-step *tq*-reduct of π. Note that, trivially, *any* occurrence of a logical rule r in π' in which some occurrence of a formula, say A, is active, stems from a *unique* occurrence r_* of (the same) logical rule in π in which an occurrence A_* of (the same up to substitution) formula is active (we call r_*, respectively A_*, the *lift* of the occurrence r, respectively A). Conversely, we define the set of *residues* of a given occurrence s of a logical rule in π as the set of all occurrences r of logical rules in π' such that $r_* = s$.

Similarly, any occurrence of a cut c in π' with cutformula A stems from a unique occurrence c_* in π with cutformula A_*, except for those occurrences of cut that are

created in an L-step (they have *no* lift in π; the reduced logical cut does *not* have a residue in π').

As a consequence, given an m-a-interspace $B_A = \sigma_0, \ldots, \sigma_n$ in π, such that σ_n's successor a is a logical or a cut rule, we can define the set of residues of B_A in π' as the set of all m-a-interspaces in π' associated with the corresponding active formulas in all residues of a in π'.

LEMMA 16 (stability lemma). *Let π be an $\mathbf{LK}^{tq}$-proof, π' be a one-step tq-reduct of π, and let B_A be an m-a-interspace of an attractive formula-occurrence A in π. If B_A is lucid (respectively w-lucid, flat) then all residues of B_A in π' are lucid (respectively w-lucid, flat).*

PROOF. Consider such a lucid m-a-interspace B_{A_s} (cf. the representation in Figure 3, where the circles indicate formula occurrences, and the vertical and diagonal lines trace corresponding formula occurrences in the block). Intuitively the lemma says, that a tq-reduction step will not change the general shape of such a block, the reason being that, in case A_s is *not* the cutformula, cuts will just pass through it in all cases, except when (i) an ancestor of the cutformula is one of the weakened formulas in the block or (ii) rule σ_0 is an axiom, and $A_{\bar{s}}$ is an ancestor of the cutformula. However, in case (i) the weakening is merely replaced by 0 or more other weakenings. In case (ii) we will be safe, thanks to A_s's attractivity. Follows the detailed argument.

First, suppose the reduction step is an L-step. If A is the (attractive) cutformula, then B_A has no residue in π'. Hence our claim vacuously holds. Otherwise, note that, by lucidity, neither of the cutformulas occurs in B_A, hence the block is untouched by the reduction (the worst that can happen is a substitution in case of a quantifier reduction step).

Next, suppose the reduction step is an S-step. If the bottom-occurrence of A is the (attractive) cutformula, let us first suppose that the step is of type S1. Then, if the top-occurrence of A is newly born in an axiom or a weakening, B_A has no residue in π'; if it is main in a logical rule, then B_A's residue is flat. If the step is of type S2 then B_A is flat (because the attractive cutformula is main in a logical rule), and so are all its residues. (Note that there are *no* residues corresponding to introductions of ancestors of the non-attractive occurrence of A by axiom or weakening.)

If A is *not* the cutformula, then, in case B_A is part of (whence, by lucidity, is properly included in) the *transported* subderivation, the residues of B_A in π' are $m \geq 0$ copies of the original block. Hence if B_A is flat or w-lucid, so are all of its residues.

Let us therefore suppose that B_A is part of (again: hence properly included in) the *transporting* subderivation. We then distinguish two cases:

- The block does *not* contain an ancestor of the cutformula. Then it will re-appear unchanged in π'.
- The block *does* contain an ancestor of the cutformula. We again distinguish two cases:
  - None of the leaves of the tree of the cutformula's ancestors is part of the block: it then will re-appear in π', modulo 'a change of context'; possible contractions on ancestors of the cutformula are replaced by 0 or more 'contextual' contractions.

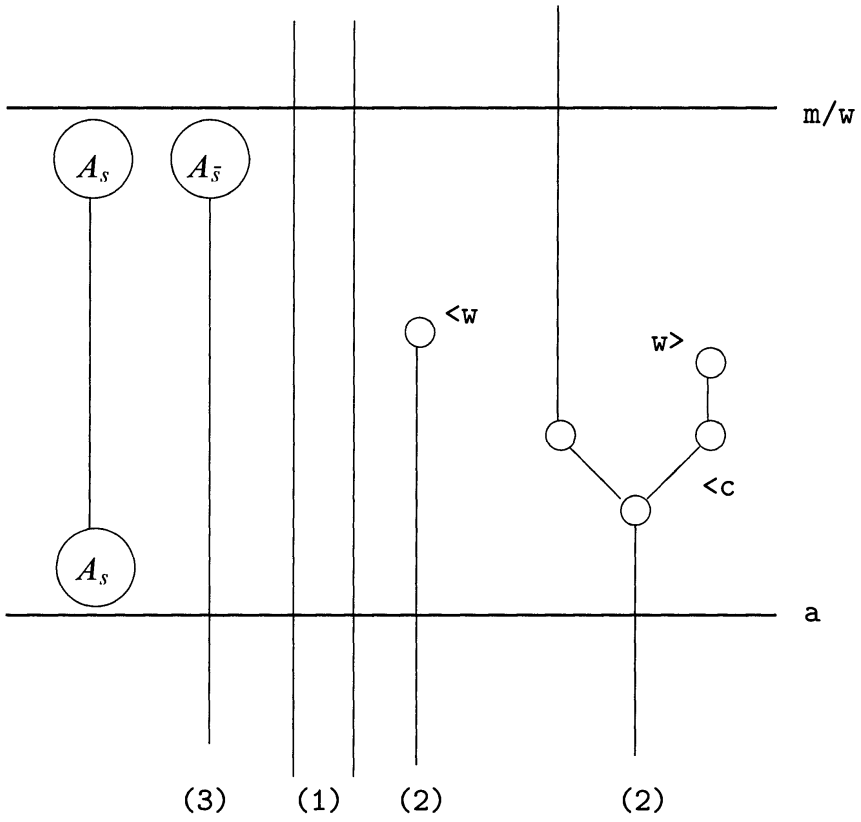


FIGURE 3. A lucid m-a-interspace B_{A_s}.

Hence the block's residue is lucid (it might even be flat); and if the original block is flat or w-lucid, then so is the residue. (Cf. the traces indicated by '(1)' in Figure 3.)

– The block does contain a leaf of the tree of the cutformula's ancestors.

⋆ If the leaf occurring in the block is a weakening, this weakening is necessarily one of the σ_i for $0 < i \leq n$ (hence B_A is not flat). After reduction, in B_A's residue, σ_i will be replaced by 0 or more 'contextual' weakenings. So the residue will be lucid. If B_A is w-lucid, then so is its residue (it might even be flat). (Cf. the traces indicated by '(2)' in Figure 3.)

⋆ If the leaf occurring in the block is an identity axiom, then it is σ_0, and because A is attractive, the reduction step under consideration is of type S2. So we are in the situation of Figure 4 (or its 'q'-dual), where r_A indicates a logical rule introducing A.

After reduction the residue of B_A remains lucid (respectively w-lucid, flat), due to the logical rule r_A. (Cf. the trace indicated by '(3)' in Figure 3.)

This exhausts all possibilities, and finishes the proof. ⊣

The reader should convince herself that the hypothesis 'attractive' is essential in the lemma above: the stronger claim, obtained by dropping the condition of attractivity, is false! There are, however, several possible variations on the theme of the stability lemma, that are of interest in other than the present applications. To mention only one: if one considers m-a-interspaces in which m is asked to be a *logical* rule, it follows (by a simple inspection of the above proof) that the condition of attractivity can be dropped.

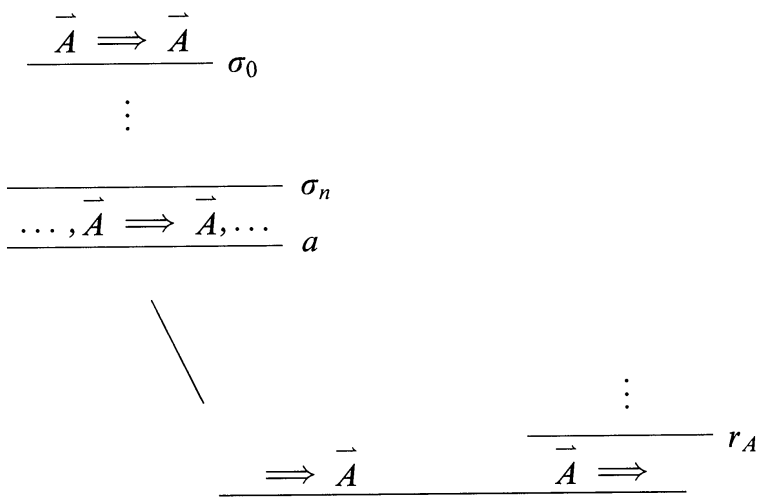


FIGURE 4. The leaf in the block is an axiom.

Observe that, if one of the m-a-interspaces B_A corresponding to an active formula occurrence A is lucid, then it is, obviously, *unique*, i.e., it is the *only* m-a-interspace associated with the given occurrence of A. Hence, by Lemma 16, if the m-a-interspace B_{A_*} corresponding to the lift A_* of a formula A, active in a logical rule r in the tq-reduct π' of a proof π, is lucid, then in π' there is a unique m-a-interspace B_A corresponding to A, which is also lucid.

We will make use of this observation when showing that several constraints on $\mathbf{LK}^{tq}$-derivations are preserved by tq-normalization, hence define computational fragments of $\mathbf{LK}^{tq}$. The most important of these is the constraint corresponding to $\mathbf{LK}^{\eta}$, defined in Section 5.5.

A remark in passing: since lucidity is preserved, one could define the energy of a cut to be less than one when the attractive cutformula has a lucid m-a-interspace, and redefine accordingly the notion of an S1, S2 and L-step. Intuitively this just says in another way that structural rules are 'delocalized', as tq-stability of weak equivalence shows.

This would allow for an instructive alternative way to prove the above lemma: we reduce it to the case that all m-a-interspaces are *w-lucid*, by using the fact that weak equivalence of derivations is tq-stable. For this it suffices to observe that given a lucid m-a-interspace in an $\mathbf{LK}^{tq}$-derivation π there is a $\pi' \overset{w}{\approx} \pi$ in which the corresponding m-a-interspace is w-lucid. This is because it is always possible to push contraction rules below a logical rule or a cut (note that in the case of a binary additive rule this may force the introduction of a weakening).

In order to conclude one needs a sharpened version of tq-stability of $\overset{w}{\approx}$, namely that whenever $\pi' \overset{w}{\approx} \pi$ and π reduces in one step to π_1 then there is a one step reduction (for the modified notion of reduction) of π' to some $\pi'_1 \overset{w}{\approx} \pi_1$.

5.4. Plug in η's! Identity sequents $A \Longrightarrow A$ are derivable by canonical derivations which we will call *η-proofs*: because the η-proof associated to $\overset{m}{\rightarrow}$ is the sequent calculus counterpart of $\lambda y^A\, x^{A \to B} y$, the η-expansion of x. Formally, they are defined as follows.

DEFINITION 17 (η-proofs). If A_i is(are) the immediate subformula(s) of A, an η-proof of $A \Longrightarrow A$ (notation: η_A), consists in the axiom(s) $A_i \Longrightarrow A_i$ and precisely one instance of each of the logical rules introducing A's main connective.

Here are some examples:

$$\dfrac{\dfrac{A \Longrightarrow A \quad B \Longrightarrow B}{A \overset{m}{\rightarrow} B, A \Longrightarrow B}}{A \overset{m}{\rightarrow} B \Longrightarrow A \overset{m}{\rightarrow} B} \qquad \dfrac{\dfrac{A \Longrightarrow A}{A, \neg A \Longrightarrow}}{\neg A \Longrightarrow \neg A} \qquad \dfrac{\dfrac{A \Longrightarrow A}{\Longrightarrow \neg A, A.}}{\neg A \Longrightarrow \neg A}$$

Note that in case A's main connective is a negation there exist precisely *two* distinct η-proofs (we will, for obvious reasons, speak of the 'intuitionistic', respectively the 'classical', $\eta_{\neg A}$).

As we will come to see, these tiny proofs are of importance.

First of all, they pop up when considering the notion of *reversibility* (of rules, formulas), well known from standard proof theory.

DEFINITION 18 (reversibility). (**R1**) A rule is said to be reversible if and only if from its conclusion one can derive its premise(s).

(**R2**) A (non-atomic) formula A_s is called reversible if and only if, whenever a sequent Γ, A_s is derivable, we can ask for A_s to be main, without loss of provability (i.e., there exists a derivation of Γ, A_s whose last rule introduces A_s's main connective).

In **LK** these notions coincide: a formula A_s is reversible if and only if its introduction rule on the s-hand side is reversible, and the reversible rules are exactly the unary multiplicative and binary additive rules, both negation rules, left existential and right universal quantifier rules.

In $\mathbf{LK}^{tq}$ all of this continues to hold:

PROPOSITION 19 (reversible rules). *Binary additive rules, unary multiplicative rules, left existential and right universal quantifier rules, as well as* both *rules for negation, are reversible. The other ones are not.*

PROOF. One just plugs (or cuts with) a piece of an η-proof associated to the connective. In the multiplicative case:

$$\dfrac{\dfrac{\vdots}{\Longrightarrow A \overset{m}{\rightarrow} B} \qquad \dfrac{A \Longrightarrow A \quad B \Longrightarrow B}{A \overset{m}{\rightarrow} B, A \Longrightarrow B}}{A \Longrightarrow B}$$

and in the additive case:

$$\dfrac{\dfrac{A \Longrightarrow A}{\Longrightarrow A, A \overset{a}{\rightarrow} B} \qquad \dfrac{\vdots}{A \overset{a}{\rightarrow} B \Longrightarrow}}{\Longrightarrow A} \qquad \dfrac{\dfrac{B \Longrightarrow B}{B \Longrightarrow A \overset{a}{\rightarrow} B} \qquad \dfrac{\vdots}{A \overset{a}{\rightarrow} B \Longrightarrow}}{B \Longrightarrow}.$$

The cases of negation and of the universal quantifier are left to the reader.

For the negative claim: it is easily verified that in general a cut free proof of an identity $A \Longrightarrow A$, that does not end with an application of a structural rule, does necessarily end with an application of a reversible rule (unless it is an axiom). ⊣

5.5. The constraint.

DEFINITION 20 ($\mathbf{LK}^{\eta}$). A proof of $\mathbf{LK}^{tq}$ is an $\mathbf{LK}^{\eta}$-proof if and only if every attractive (occurrence of a) formula active in an irreversible or a negation rule is main.

This constraint can be rephrased as follows: the m-a-interspace associated to an attractive formula occurrence active in an irreversible or a negation rule is flat. Thus, the stability lemma of the previous section implies almost directly that $\mathbf{LK}^{\eta}$ indeed is a computational fragment of $\mathbf{LK}^{tq}$.

THEOREM 21. $\mathbf{LK}^{\eta}$ *is stable under tq-normalization.*

PROOF. Let $\pi \in \mathbf{LK}^{\eta}$, and suppose π tq-reduces to π'. Suppose A is an attractive formula occurrence active in an irreversible or negation rule r in π', and r is a residue of an occurrence r_* of the same rule in π. The corresponding active occurrence A_* in r_* is attractive, hence (by hypothesis), the (unique) m-a-interspace associated with A_* is a flat block B_{A_*} in π. By Lemma 16 all residues of B_{A_*} are flat, in particular the block B_A corresponding to our chosen formula occurrence A in π'. Hence A is main. ⊣

As expected, we no longer need an orientation of multiplicative connectives, nor two versions of the unary multiplicative rules, for the output of the (q,t) L-step for the implication will be:

$$\dfrac{\dfrac{}{\overset{\leftarrow}{\Longrightarrow} A}\,\mathrm{R}_A \qquad \dfrac{\vdots}{\overset{\leftarrow}{A} \Longrightarrow \vec{B}} \qquad \dfrac{}{\vec{B} \Longrightarrow}\,\mathrm{R}_B}{\Longrightarrow}$$

where A on the rhs and B on the lhs are *main*, so that the order of the cuts is irrelevant with respect to tq-normalization (Lemma 4).

Moreover, with respect to provability, nothing is lost.

THEOREM 22 (completeness). *All sequents provable in* $\mathbf{LK}^{tq}$ *are provable in* $\mathbf{LK}^{\eta}$.

PROOF. This is because *un*constrained rules are *derivable* with the help of cuts with pieces of η-proofs in $\mathbf{LK}^{\eta}$ (whence its name). We only deal with implications and, as usual, do not represent the contexts. "Would-be active" formulas below are denoted by $\bar{A}$.

In the multiplicative case:

$$\dfrac{\dfrac{\vdots}{\Longrightarrow \bar{A}} \qquad \dfrac{A \Longrightarrow A \quad B \Longrightarrow B}{A, A \overset{m}{\rightarrow} B \Longrightarrow B} \qquad \dfrac{\vdots}{\bar{B} \Longrightarrow}}{A \overset{m}{\rightarrow} B \Longrightarrow}\,.$$

In the additive case:

$$\dfrac{\dfrac{A \Longrightarrow A}{\Longrightarrow A \overset{a}{\rightarrow} B, A} \quad \vdots \quad \bar{A} \Longrightarrow}{\Longrightarrow A \overset{a}{\rightarrow} B} \qquad \dfrac{\vdots \quad \dfrac{B \Longrightarrow B}{B \Longrightarrow A \overset{a}{\rightarrow} B}}{\Longrightarrow A \overset{a}{\rightarrow} B} \quad \Longrightarrow \bar{B}$$

For the universal quantifier:

$$\dfrac{\dfrac{Ay \Longrightarrow Ay}{\forall x\, Ax \Longrightarrow Ay} \quad \vdots \quad \bar{A}y \Longrightarrow}{\forall x\, Ax \Longrightarrow}$$

In case of negation:

$$\dfrac{\vdots \quad \Longrightarrow \bar{A} \qquad \dfrac{A \Longrightarrow A}{A, \neg A \Longrightarrow}}{\neg A \Longrightarrow} \qquad \dfrac{\dfrac{A \Longrightarrow A}{\Longrightarrow \neg A, A} \quad \vdots \quad \bar{A} \Longrightarrow}{\Longrightarrow \neg A}$$

All these transformations (made explicit only if one eliminates the cuts introduced) are identities if the would-be active formula is (1) not attractive or (2) already main, i.e., in the case where it needs no correction.

In fact we have better than a completeness theorem: the proof above shows how to *project* $\mathbf{LK}^{tq}$ onto $\mathbf{LK}^{\eta}$ by eliminating cuts. By the preceding remark this projection leaves $\mathbf{LK}^{\eta}$ pointwise invariant up to trivial structural steps.

Observe that in the multiplicative case the order of the cuts matters when both would-be active formulas are attractive. So there are two possible ways to project a dilemmatic multiplicative rule in $\mathbf{LK}^{\eta}$.

In the additive case the derivation of the unconstrained rule is not intuitionistic, so that only the constraint on $\mathbf{LJ}^{q}$ would be complete for **LJ**-provability with respect to the additive implication. Anyway **LJ** with an additive implication is itself not complete for intuitionistic truth; that is why there is no additive intuitionistic implication.

In **LJ** there are $O(2^n)$ cut-free ways to prove X, $[X \overset{m}{\rightarrow} X]^n \Longrightarrow X$ (where $[X \overset{m}{\rightarrow} X]^n$ stands for n occurrences of $X \overset{m}{\rightarrow} X$), without using structural rules; there is, however, only *one* way to do this in the constrained system $\mathbf{LJ}^{\eta}$, a sobriety alien to usual sequent calculi (as to old American cars). Sobriety shows up again in the fact that only *one* of the two η-proofs of $\neg(A^{\varepsilon}) \Longrightarrow \neg(A^{\varepsilon})$ associated to negation is correct with respect to the constraint: the 'intuitionistic' one if ε is q, the 'classical' one in case of t. Hence in $\mathbf{LK}^{\eta}$ *all* η-proofs are unique. This has an interesting consequence for the notion of *reversibility* in $\mathbf{LK}^{\eta}$: Proposition 19 continues to hold for $\mathbf{LK}^{\eta}$, but we have to replace reversible by $R1$-reversible (in the sense of Definition 18). In the case of negation, due to the fact that attractive negated formulas in $\mathbf{LK}^{\eta}$ are asked to be main, the notions of $R1$- and $R2$-reversibility no longer coincide: negation in $\mathbf{LK}^{\eta}$ is always $R1$-reversible, but $(\neg A^{\varepsilon})_s$ is $R2$-reversible if and only if $A^{\varepsilon}_{\bar{s}}$ is non-attractive (one cannot have both the moon and the money for the moon).

5.6. $\mathbf{LK}^{\eta}$ and linear logic. The constraint characterizing $\mathbf{LK}^{\eta}$-derivations, is stronger than strictly necessary in order to recover commutation of cuts in the 'logical dilemma'. As we know by Lemma 4, to that effect it would suffice to ask that *at least one* of the attractive (occurrences of) formulas active in an irreversible multiplicative rule is main. Again by Lemma 16, this constraint defines a computational fragment (say $(\mathbf{LK}^{\eta})'$) of $\mathbf{LK}^{tq}$, which, as $\mathbf{LK}^{\eta}$, is complete for $\mathbf{LK}^{tq}$-provability. Contrary to $\mathbf{LK}^{\eta}$, however, the stringent embedding which we are going to define is *not* a decoration for this fragment,[8] and it is this embedding, giving a *very* tight fit between $\mathbf{LK}^{\eta}$ and **LL**, which in the end justifies our choosing the more restricted system.

DEFINITION 23 (stringo translation). Take ε, ε' in $\{t, q\}$, we define a mapping S of coloured classical formulas to linear formulas: $S(X^{\varepsilon}) = X$, and

$$S(A^{\varepsilon} \xrightarrow{m} B^{\varepsilon'}) = \mu_L^{\varepsilon} S(A) \multimap \mu_R^{\varepsilon'} S(B) \qquad S(A^{\varepsilon} \xrightarrow{a} B^{\varepsilon'}) = \mu_R^{\varepsilon} S(A) \rightsquigarrow \mu_L^{\varepsilon'} S(B)$$

$$S(\forall X\, A^{\varepsilon}) = \forall X\, \mu_R^{\varepsilon} S(A)$$

$$S(\neg A^{t}) = (\mu_R^{t} S(A))^{\perp} \qquad S(\neg A^{q}) = (\mu_L^{q} S(A))^{\perp}.$$

Explicitly, S is defined by:

$A \xrightarrow{m} B$	t	q
t	$!?A \multimap ?B$	$!?A \multimap ?!B$
q	$!A \multimap ?B$	$!A \multimap ?!B$

$A \xrightarrow{a} B$	t	q
t	$?A \rightsquigarrow !?B$	$?A \rightsquigarrow !B$
q	$?!A \rightsquigarrow !?B$	$?!A \rightsquigarrow !B$

$\forall XA$	
t	$\forall X?A$
q	$\forall X?!A$

$\neg A$	
t	$(?A)^{\perp}$
q	$(!A)^{\perp}$

As in the pletho-case, again it suffices to deal with implications, negation and universal quantification, as the treatment of the other connectives and the existential quantifier follows by 'duality', cf. the remark on page 769.

This translation is entirely determined by asking for a decoration of $\mathbf{LK}^{\eta}$ with a void pre-modalization for ($R2$-) reversible formulas.

THEOREM 24 (stringo embedding). *The translation of Definition* 23 *above allows for an inductive decoration mapping proofs of* $\mathbf{LK}^{\eta}$ *to proofs of* **LL**.

PROOF. A proof π of a sequent $\Gamma^{t}, \Lambda^{q} \Longrightarrow \Delta^{t}, \Sigma^{q}$ is mapped to a proof $S(\pi)$ of $!?S(\Gamma), !S(\Lambda) \Longrightarrow ?S(\Delta), ?!S(\Sigma)$, but now we take into account the fact that attractive formulas active in an irreversible rule are main.

Again, as in the proof of Theorem 7, we want to prefix the active formulas in the premises of all rules for a given connective in such a way that we get the same final modalities for these formulas in all cases. But this time we demand that these prefixes in a reversible rule for a binary connective or quantifier be void (as already mentioned, for these reversible rules there is no pre-modalization, only a post-modalization)! Now in general in $\mathbf{LK}^{tq}$ this is problematic in case an active formula in the reversible rule is non-attractive: it then, in the inductive decoration

[8] And $(\mathbf{LK}^{\eta})'$ has the funny property that its optimal decoration is non-deterministic, while nevertheless its normalization is fully determined.

procedure, has obtained a single modal prefix (! or ?), while the corresponding formula in an irreversible rule has a double modal prefix (?! or !?). Hence, for this to work, we have to *remove* the outermost exponential. However, being in $\mathbf{LK}^{\eta}$, we know that this formula, active in the irreversible rule, is *attractive*, hence is *main*: it has just been introduced in an axiom or a logical rule, whence our problem is simply solved by not applying a dereliction before application of the irreversible rule precisely in those cases. We leave it to the reader to provide the (similar) argument for the negation rules. ⊣

The fact that $\mathbf{LK}^{\eta}$ dodges the logical dilemma is here mirrored in the fact that (using the notation of the proof of Theorem 7) we can take $\Box_A$ and $\Box_B$ to be void, whence the 'unification problem' in case of the unary rules has become trivial and there no longer are two symmetric solutions in the (q, t)-case.

PROPOSITION 25 (stringo or S-semantics). *Let π be a proof of* $\mathbf{LK}^{\eta}$, *and let $S(\pi)^*$ be the denotation of $S(\pi)$ in linear logic's coherent semantics. This interpretation is non-trivial and invariant under tq-normalization (in other words, $\star \circ S$ is a denotational semantics for* $\mathbf{LK}^{\eta}$).

PROOF. The proof will proceed as in the case of the D-semantics provided we know that the proof net associated $S(\pi)$ and adjusted as usual will simulate the tq-steps. But that proof net is just $D^{-}(\pi)$ minus superfluous exponentials and hence will reduce exactly as $D^{-}(\pi)$ up to dereliction and commutative steps. ⊣

Let us take a moment to indulge in a brief heuristical and historical flash-back, by tracing the path that originally lead us to $\mathbf{LK}^{\eta}$, a path that stays inside linear logic (cf. our [4]). The proof by induction on the length of derivations in intuitionistic sequent calculus of the correctness of Girard's original translation of intuitionistic logic in **LL**, where the intuitionistic arrow is read as $!A \multimap B$, relies on a proof of $!(!A \multimap B) \Longrightarrow !A \multimap !B$.

This translation is an early example of what in this paper we call a stringent translation. From the point of view of sequent calculus reading the intuitionistic arrow as $!A \multimap !B$, an example of a pletho translation, is way simpler. But sparing exponentials amounts to building a computational twin to our proof that is more flexible than the pletho one, more flexible here meaning: less exponential boxes, hence more sequentializations, hence a larger tq-stable notion of equivalence of proofs induced by the translation. So, in order to get a 'high fidelity' classical syntax, it was very natural and tempting to try the same trick (economizing exponentials) also in the case of classical implication.

Well, let us then prove that the translation of Definition 23 is *sound* with respect to $\mathbf{LK}^{tq}$ provability (the indirect proof says that this is a corollary to Theorem 24, because of Theorem 22).

The idea is, to just inductively apply the stringo translation to an (unconstrained) $\mathbf{LK}^{tq}$-derivation. This is straightforward for the reversible rules, which are translated by the corresponding (reversible) **LL**-rule followed by the appropriate post-modalization.

In case of an irreversible rule, we apply the *pletho* pre-modalization to the active formulas, and then apply the corresponding (irreversible) **LL**-rule. Now the resulting formula $(\frac{D}{S}(X))_s$ (D on the surface, S below),[9] of course will in general *not* be the stringo translation $S(X)$ of the $\mathbf{LK}^{tq}$-formula X. If it is not, in order for it to become so, we have to remove one or more exponentials. This we achieve by cutting with a sequent $(\mu_s^{\varepsilon} S(X))_s, (\frac{D}{S}(X))_{\bar{s}}$. We refer to such a sequent as a *constrictive morphism.*

Hence irreversible rules are translated in the following way:

$$\frac{\text{pletho pre-modalization of the active formulas}}{\dfrac{\mathbf{LL}\text{ associated rule}}{\text{associated c. m.}}}.$$

Let us list the constrictive morphisms needed for the implications, leaving the cases of negation and universal quantification to the reader.

For the multiplicative style, we need:

A/B	t	q
t	$![?]\{!?A \multimap ?B\} \Longrightarrow !?A \multimap !?B$	
q	$![?]\{!A \multimap ?B\} \Longrightarrow !?!A \multimap ?!?B$	$![?]\{!A \multimap ?!B\} \Longrightarrow ?!A \multimap ?!B$

where $[\cdot]$ means that the modality inside is optional (so the morphism succeeds whatever the colour of the implication). For the additive style, we need:

A/B	t	q
t	$!?A \overset{a}{\rightsquigarrow} !?B \Longrightarrow ?[!]\{?A \overset{a}{\rightsquigarrow} !?B\}$	$!?A \overset{a}{\rightsquigarrow} ?!B \Longrightarrow ?[!]\{?A \overset{a}{\rightsquigarrow} !B\}$
q		$?!A \overset{a}{\rightsquigarrow} ?!B \Longrightarrow ?[!]\{?!A \overset{a}{\rightsquigarrow} !B\}$

Now all this would be pretty worthless if all, or even some, of these sequents were not derivable. However, that is not the case.[10]

LEMMA 26. *All constrictive morphisms are derivable.*

PROOF. We treat only the case of the binary connectives. Let $\bullet$ denote such a connective. If the inductive application of the stringo translation as defined above results in a formula $\frac{D}{S}(A \bullet B)$ containing too many exponentials, this formula is introduced in an *irreversible* rule. In the constrictive morphism then $\frac{D}{S}(A \bullet B)$ occurs at the *reversible* side.

Now take an η-proof of $S(A \bullet B) \Longrightarrow S(A \bullet B)$. Recall that the last rule applied in this proof is the reversible rule. Consider the proof minus this last rule. Because the exponentials now prefixing the 'freed' subformulas are of type '?' (since components of the stringo translation of a reversible formula always are), we can *add* exponentials to obtain $\mu_s^{\varepsilon} S(A \bullet B)$, then *add* the appropriate exponentials $\Box_A$, $\Box_B$ to $S(A)$ and $S(B)$ and apply the rule in order to obtain $\frac{D}{S}(A \bullet B)$. ⊣

The above construction generated a set of $\mathbf{LK}^{tq}$-proofs, complete for classical provability, and by its uniformity hinted at the existence of a complete subsystem

[9] Sous les pavés, la plage.

[10] For the record: we had a jolly good time deriving specific examples of constrictive morphisms using Tanel Tammet's **LL**-theorem-prover, cf. [42].

of $\mathbf{LK}^{tq}$. Therefore it was only natural to wonder about the structure of **LL**-derivations of S-translated classical sequents, which we will call S-sequents.

DEFINITION 27 (S-proofs). A sequent σ of linear formulas is said to be an *S-sequent* if it is the stringo translation of an $\mathbf{LK}^{tq}$-sequent, i.e., if it is of the form

$$!?S(\Gamma), !S(\Lambda) \Longrightarrow ?S(\Delta), ?!S(\Sigma).$$

It is said to be a *sub-S-sequent* if all formulas occurring are subformulas of stringo translations of $\mathbf{LK}^{tq}$-formulas, i.e., if it is of the form

$$S(\Theta), ?S(\Theta'), ?!S(\Theta''), !?S(\Gamma), !S(\Lambda) \Longrightarrow ?S(\Delta), ?!S(\Sigma), !?S(\Psi''), !S(\Psi'), S(\Psi).$$

Similarly we can speak of S-proofs meaning that all sequents occurring are sub-S-sequents.

This is what we found:

LEMMA 28 ($\beth$-lemma). *Take a sub-S-sequent σ:*

$$S(\Theta), ?S(\Theta'), ?!S(\Theta''), !?S(\Gamma), !S(\Lambda) \Longrightarrow ?S(\Delta), ?!S(\Sigma), !?S(\Psi''), !S(\Psi'), S(\Psi)$$

and define

$$\beth(\sigma) = |\Theta', \Theta'', \Psi'', \Psi'| + \tfrac{1}{2}\,|\,\mathrm{atoms}(\Theta, \Psi)| + |\,\mathrm{IRR}(\Theta, \Psi)|,$$

where $\mathrm{atoms}(\Theta, \Psi)$ *is the multiset of atoms occurring in* Θ, Ψ, *and* $\mathrm{IRR}(\Theta, \Psi)$ *the multiset of the irreversible formulas in* Θ, Ψ. *If σ is provable in* **LL**, *then* $\beth(\sigma) \leq 1$ *and any S-proof of σ will wholly consist in sequents σ' with* $\beth(\sigma') \leq 1$.

PROOF. By induction on derivations in **LL**. In case of the rules for the binary connectives, use the fact that components of a *reversible* formula $S(A)$ are of type '?', those of an *irreversible* one are of type '!'. ⊣

Note however that the constraint described by the lemma is not nearly as strict as the one imposed upon $\mathbf{LK}^{\eta}$-derivations. If one would like to get closer to $\mathbf{LK}^{\eta}$, one may restrict the form of linear derivations of S-sequents still further.

DEFINITION 29 ($\beth'$, $\mathbf{LL}^{-}$). Take an **LL**-sequent σ:

$$\Gamma, !\Lambda \Longrightarrow ?\Delta, \Sigma$$

where no formula in Γ (respectively Σ) begins with a '!' (respectively '?'), and define $\beth'(\sigma) = |\Gamma, \Sigma|$. By $\mathbf{LL}^{-}$ we mean **LL** restricted to sequents σ with $\beth'(\sigma) \leq 1$.

That is, $\mathbf{LL}^{-}$ (which, by the way, is easily seen to be closed under a reasonable sequent version of cut-elimination) is restricted to the use of sequents in which there is *at most one* formula which is not of type '?'.

Then any S-proof in $\mathbf{LL}^{-}$ is, modulo some negligible permutations of structural rules, the decoration of an $\mathbf{LK}^{\eta}$-proof.

$$\frac{\ldots, A_s, B_{s'}, \ldots}{\ldots, A_s \bullet_m B_{s'}, \ldots} \quad \xrightarrow{\alpha} \quad \frac{\dfrac{\dfrac{\ldots, A_s, B_{s'}, \ldots}{\ldots, A_s \bullet_a B_{s'}, B_{s'}, \ldots}}{\ldots, A_s \bullet_a B_{s'}, A_s \bullet_a B_{s'}, \ldots}}{\ldots, A_s \bullet_a B_{s'}, \ldots}$$

$$\frac{\Gamma, A_s \qquad \Delta, B_{s'}}{\Gamma, \Delta, A_s \bullet_m B_{s'}} \quad \xrightarrow{\alpha} \quad \frac{\dfrac{\Gamma, A_s}{\vdots} \qquad \dfrac{\Delta, B_{s'}}{\vdots}}{\Gamma, \Delta, A_s \bullet_a B_{s'}}$$

$$\frac{\ldots, A, \ldots}{\ldots, A \bullet_a B, \ldots} \quad \xrightarrow{\mu} \quad \frac{\dfrac{\ldots, A, \ldots}{\ldots, A, B, \ldots}}{\ldots, A \bullet_m B, \ldots}$$

$$\frac{\ldots, B, \ldots}{\ldots, A \bullet_a B, \ldots} \quad \xrightarrow{\mu} \quad \frac{\dfrac{\ldots, B, \ldots}{\ldots, A, B, \ldots}}{\ldots, A \bullet_m B, \ldots}$$

$$\frac{\Gamma, A_s \qquad \Gamma, B_{s'}}{\Gamma, A_s \bullet_a B_{s'}} \quad \xrightarrow{\mu} \quad \frac{\dfrac{\Gamma, A_s \qquad \Gamma, B_{s'}}{\Gamma, \Gamma, A_s \bullet_m B_{s'}}}{\vdots}$$
$$\Gamma, A_s \bullet_m B_{s'}$$

FIGURE 5. Additivation α, and multiplicativation μ.

5.7. Plug in θ's! As already observed in the preliminary remarks of Section 2.2 the additive and multiplicative formulations of the rules for the binary connectives are *interderivable* by means of the structural rules. This interderivability can be used to define two operators, say α and μ, which act on an $\mathbf{LK}^{(tq)}$-proof π by replacing additive connectives occurring in formulas introduced in an axiom or by weakening by their multiplicative analogues and replacing *all* occurrences of multiplicative rules by their additive simulation, respectively *all* occurrences of additive rules by their multiplicative simulation. The precise actions of α and μ on occurrences of rules for the connectives $\bullet_m$ and $\bullet_a$ are depicted in Figure 5.

DEFINITION 30 (θ-proofs). A θ-proof of a sequent $A \bullet_\sigma B \Longrightarrow A \bullet_\sigma B$, is the derivation obtained by applying α (in case $\sigma = a$), or μ (in case $\sigma = m$) to the η-proof of $A \bullet_{\bar{\sigma}} B \Longrightarrow A \bullet_{\bar{\sigma}} B$. I.e., $\theta_{A \bullet_a B} = \alpha(\eta_{A \bullet_m B})$, $\theta_{A \bullet_m B} = \mu(\eta_{A \bullet_a B})$.

As an example, here is $\theta_{A \overset{m}{\to} B}$:

$$\dfrac{\dfrac{\dfrac{A \Longrightarrow A}{A \Longrightarrow A, B}}{\Longrightarrow A, A \to B} \qquad \dfrac{\dfrac{B \Longrightarrow B}{A, B \Longrightarrow B}}{B \Longrightarrow A \to B}}{A \to B \Longrightarrow A \to B, A \to B}$$
$$A \to B \Longrightarrow A \to B$$

In fact, the θ-proofs show us precisely how far we can go in *approximating* reversion of the irreversible rules: plugging θ's, we find that whenever π is a derivation of Γ, $(A \bullet B)_s$, and the introduction rule R for $\bullet$ is *irreversible*, then there is a derivation π' of Γ, $(A \bullet B)_s$ whose last *logical* rule is R. In fact, R will be the last rule of π' *modulo a certain number* ≥ 0 *of contractions.*

Hence irreversible binary connectives might be baptized quasi-reversible! The dual of this "quasi-reversion" induces another possible constraint on $\mathbf{LK}^{tq}$-proofs.

DEFINITION 31 ($\mathbf{LK}^{\theta}$). A proof of $\mathbf{LK}^{tq}$ is an $\mathbf{LK}^{\theta}$-proof if and only if every attractive (occurrence of a) formula active in a reversible rule for a binary connective is newly born (that is, is either main or has just been weakened), modulo some number of weakenings (i.e., if and only if the associated m-a-interspace is w-lucid).

DEFINITION 32. $\mathbf{LK}^{\eta\theta} := \mathbf{LK}^{\eta} \cap \mathbf{LK}^{\theta}$.

THEOREM 33. $\mathbf{LK}^{\theta}$ *and* $\mathbf{LK}^{\eta\theta}$ *are stable under tq-normalization.*

PROOF. Similar to the proof of Theorem 21, now using preservation of w-lucidity.⊣

THEOREM 34. *All sequents provable in* $\mathbf{LK}^{tq}$ *are provable in* $\mathbf{LK}^{\theta}$, $\mathbf{LK}^{\eta\theta}$.

PROOF. Yep, indeed, again this is because unconstrained rules are derivable with the help of cuts in $\mathbf{LK}^{\theta}$: plug in the θ's!

In the multiplicative case:

$$\dfrac{\dfrac{\dfrac{A \Longrightarrow A}{A \Longrightarrow A, B}}{\Longrightarrow A, A \overset{m}{\rightarrow} B} \qquad \begin{matrix}\vdots \\ \bar{A} \Longrightarrow \bar{B}\end{matrix} \qquad \dfrac{\dfrac{B \Longrightarrow B}{B, A \Longrightarrow B}}{B \Longrightarrow A \overset{m}{\rightarrow} B}}{\dfrac{\Longrightarrow A \overset{m}{\rightarrow} B, A \overset{m}{\rightarrow} B}{\Longrightarrow A \overset{m}{\rightarrow} B}}$$

And for the additive one:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \Longrightarrow \bar{A}, \Delta\end{matrix} \qquad \dfrac{\dfrac{A \Longrightarrow A}{A \Longrightarrow A, B} \quad \dfrac{B \Longrightarrow B}{A, B \Longrightarrow B}}{A, A \overset{a}{\rightarrow} B \Longrightarrow B} \qquad \begin{matrix}\vdots \\ \Gamma, \bar{B} \Longrightarrow \Delta\end{matrix}}{\Gamma, \Gamma, A \overset{a}{\rightarrow} B \Longrightarrow \Delta, \Delta}}{\begin{matrix}\vdots \\ \Gamma, A \overset{a}{\rightarrow} B \Longrightarrow \Delta\end{matrix}}$$

(Note that, in case π is η-correct, it remains so after plugging of θ's; in case π is θ-correct, it remains so after plugging of η's. The two constraints are independent.)⊣

Unlike for the irreversible rules observe that the 'projections' of Proposition 34 are not 'blank' when applied to non-attractive formula-occurrences, or to w-lucid attractive formula-occurrences!

It seems not to be possible to introduce a similar constraint for the reversible quantifier-rules. No wonder, as the effect of plugging θ's boils down to a 'pseudo-change' of style: the *reversible* rule for $\bullet_\sigma$ is replaced by, say, the σ-simulation of the *irreversible* rule for $\bullet_{\bar{\sigma}}$. Such a 'style'-change does not exist for the quantifiers.

The calculi $\mathbf{LK}^\theta$ and $\mathbf{LK}^{\eta\theta}$ can be decorated with interesting refinements of the pletho-, respectively the stringo-translation: the θ-constraint enables us to replace some of the exponentials occurring in these translations by *weakening-exponentials* $!_w$, respectively $?_w$. (In fact, these exponentials are internally definable in **LL**: $!_w A \equiv \mathbf{1}\&A$, $?_w A \equiv \bot\oplus A$.) A formula prefixed by such an exponential cannot be subjected to *contraction*. For details on systems of linear logic with several types of exponentials see [3]. Using the terminology of that paper, the following translations map $\mathbf{LK}^{(\eta)\theta}$ into a bi-colour linear logic, with $!/? > !_w/?_w$ (a $!/?$-promotion is not possible in a context containing formulas prefixed by a weak exponential).

DEFINITION 35 (pletho$^\theta$ translation). We define inductively a mapping D^θ of coloured classical formulas to linear formulas by $D^\theta(X^t) = D^\theta(X^q) = X$ for atoms, and by the following tables:

$A \overset{m}{\to} B$	t	q
t	$!?A \multimap !?B$	$!_w?A \multimap ?_w!B$
q	$!?!A \multimap ?!?B$	$?!A \multimap ?!B$

$A \overset{a}{\to} B$	t	q
t	$!?A \rightsquigarrow !?B$	$!?A \rightsquigarrow ?!B$
q	$?!_w A \rightsquigarrow !?B$	$?!_w A \rightsquigarrow ?!B$

$\forall X\, A$	
t	$\forall X\, !?A$
q	$\forall X\, ?!A$

$\neg A$	
t	$(!?A)^\perp$
q	$(?!A)^\perp$

THEOREM 36. *The mapping of Definition* 35 *allows for a decoration of* $\mathbf{LK}^\theta$.

PROOF. As the proofs for $\mathbf{LK}^{tq}$, $\mathbf{LK}^\eta$. And of course using the θ-constraint. Contextual modalities are *full permission*, because of the clauses in the definition that during decoration ask for a 'full-permission' promotion. ⊣

DEFINITION 37 (stringo$^\theta$ translation). We define inductively a mapping S^θ of coloured classical formulas to linear formulas by $S^\theta(X^\varepsilon) = X$ for atoms, and by the following tables:

$A \overset{m}{\to} B$	t	q
t	$!_w?A \multimap ?B$	$!_w?A \multimap ?_w!B$
q	$!A \multimap ?B$	$!A \multimap ?_w!B$

$A \overset{a}{\to} B$	t	q
t	$?A \rightsquigarrow !_w?B$	$?A \rightsquigarrow !B$
q	$?!_w A \rightsquigarrow !_w?B$	$?!_w A \rightsquigarrow !B$

$\forall X\, A$	
t	$\forall X\, ?A$
q	$\forall X\, ?!A$

$\neg A$	
t	$(?A)^\perp$
q	$(!A)^\perp$

THEOREM 38. *The mapping of Definition* 37 *allows for a decoration of* $\mathbf{LK}^{\eta\theta}$.

§6. Polaro embedding. In this section, we equip the *polarized* fragment of $\mathbf{LK}^\eta$, denoted by $\mathbf{LK}^\eta_p$ and defined right below, with a stronger tq-stable equivalence relation based upon the reversibility properties of classical connectives (cf. Proposition 19). As a by-product, we may introduce an alternative noetherian and confluent

normalization which is shown to compute the same normal form as tq. Next, as in the stringo case, we trace back the path leading to that equivalence by exhibiting an even more energy-sparing translation in **LL** that generates it. This so-called *polaro* translation is in fact Girard's translation devised for **LC** which trivially extends to $\mathbf{LK}^{\eta}_{p}$.

DEFINITION 39 (polarized proofs). An $\mathbf{LK}^{tq}$-proof is said to be *polarized* if a non-atomic formula occurrence is reversible if and only if it is non-attractive.

In other words, occurrences of right (respectively left) reversible binary connectives and quantifiers are coloured t (respectively q), and negation rules preserve attractivity, or equivalently flip colours (i.e., A^{ε} if and only if $(\neg A)^{\bar{\varepsilon}}$). Because: A_s is attractive if and only if $(\neg A_s)_{\bar{s}}$ is irreversible which is the case if and only if it is attractive by polarization. Since negation flips sides, in order to preserve attractivity, it has to flip colours as well. Intuitively this just means that polarized negation will be readable in **LL** as a mere change of side, i.e., as a real involutive construct.

DEFINITION 40 ($\mathbf{LK}^{\eta}_{p} \subset \mathbf{LK}^{\eta}$). An $\mathbf{LK}^{\eta}$-proof is in $\mathbf{LK}^{\eta}_{p}$ if and only if it is polarized.

Commutation of reversible rules is invariant with respect to the S-semantics in $\mathbf{LK}^{\eta}_{p}$. Indeed, by the η-constraint, reversible rules are translated by the corresponding **LL**-rules plus a post-modalization which depends on the colour of the main formula. Saying a proof is polarized is exactly saying that these post-modalizations always are just derelictions. But the difference in order in two blocks [reversible rule/dereliction] is semantically invisible in **LL**.

Also, there is the *antipolarized* fragment of $\mathbf{LK}^{\eta}$, say $\mathbf{LK}^{\eta}_{ap}$.

DEFINITION 41 (antipolarized proofs). An $\mathbf{LK}^{tq}$-proof is said to be *antipolarized* if a non-atomic formula occurrence is reversible if and only if it is attractive.

This time negation rules preserve colours or equivalently flip attractivity. Let η be an antipolarized η-proof, then $S(\eta)^{\star}$ is the identity. Since, now the irreversible rules get a mere dereliction as post-modalization. Just commute that dereliction and the reversible rule below in $S(\eta)$ (which is easily seen to be invisible semantically, maybe a little less than easily in the additive case) and what we find is an *expansion* (see the definition below) of an η-proof of **LL**, hence is interpreted by a semantic identity (for the linear denotation of all expanded η-proofs is the identity).

Since attractive non-atomic formulas are reversible in $\mathbf{LK}^{\eta}_{ap}$, there is another way to constrict irreversible rules and map antipolarized $\mathbf{LK}^{tq}$-proofs to $\mathbf{LK}^{\eta}$, namely to reverse their attractive non-atomic active formulas, which leads to different constricted forms. Contrariwise, we lose that option in constricting polarized $\mathbf{LK}^{tq}$-proofs, which is an indication of $\mathbf{LK}^{\eta}_{p}$'s sobriety.

Let A be an occurrence of a non-atomic formula in a proof π. The *expansion of* A *in* π yields a proof π^{e}, which is obtained by replacing in π all axiom rules introducing an ancestor of A by η_A.

Let π be a proof of $\Gamma \Longrightarrow \Delta, A \overset{m}{\rightarrow} B$. The *reversion of* $A \overset{m}{\rightarrow} B$ *in* π yields a proof π^{r}, which is obtained by:

1. expanding $A \stackrel{m}{\to} B$;
2. erasing (necessarily right) implication rules whose main formulas are ancestors of $A \stackrel{m}{\to} B$;
3. transferring structural rules on ancestors of $A \stackrel{m}{\to} B$ to their components A and B;
4. adding as a last rule the implication rule to obtain $A \stackrel{m}{\to} B$.

Let π be a proof of $\Gamma, A \stackrel{a}{\to} B \Longrightarrow \Delta$. The *reversion of* $A \stackrel{a}{\to} B$ *in* π yields a proof π^r, which is obtained by:

1. expanding $A \stackrel{a}{\to} B$;
2. projecting π along A, that is to say erasing (necessarily left) implication rules whose main formulas are ancestors of $A \stackrel{a}{\to} B$ and their premise containing the active B; then transferring structural rules on ancestors of $A \stackrel{a}{\to} B$ to A;
3. projecting π along B as well;
4. adding as last rule between the resulting two projections above a left implication rule in order to get $A \stackrel{a}{\to} B$.

Of course reversion is precisely what happens when one eliminates the cut introduced in the proof of reversibility (Proposition 19). And there are similar reversions for each connective and for each of the quantifiers: the formula occurrence to be reversed must be ... reversible. Reversion can be internalized, as can be shown by careful computation, as long as the formula to be reversed is polarized.

LEMMA 42 (internal reversion). *Let* π_0 *be a proof of* $\mathbf{LK}^{tq}$, *then*:

$$\dfrac{\begin{array}{cc}\pi_0 & \eta \\ \vdots & \vdots \\ \Gamma \Longrightarrow \Delta, A \stackrel{m}{\to} B & A \stackrel{m}{\to} B \Longrightarrow A \stackrel{m}{\to} B\end{array}}{\Gamma \Longrightarrow \Delta, A \stackrel{m}{\to} B}$$

tq-reduces to π_0^r *when* $A \stackrel{m}{\to} B$ *is* t *and* π_0^e *when* $A \stackrel{m}{\to} B$ *is* q.

Similar internalizations may be computed for additives and quantifiers. In the case of negation something expected happens:

$$\dfrac{\begin{array}{cc}\pi & \eta_{(\neg A^{\varepsilon})^{\varepsilon'}} \\ \vdots & \vdots \\ \Gamma \Longrightarrow \Delta, (\neg A^{\varepsilon})^{\varepsilon'} & (\neg A^{\varepsilon})^{\varepsilon'} \Longrightarrow (\neg A^{\varepsilon})^{\varepsilon'}\end{array}}{\Gamma \Longrightarrow \Delta, (\neg A^{\varepsilon})^{\varepsilon'}}$$

reduces to π^r if and only if ε is q and ε' is t; so that only reversion of colour-'flipping' negations can be internalized; moreover the reversion has to take place on the side where the negation is reversible (right (respectively left) side if ε is q (respectively t)).

LEMMA 43. *Consider in* $\mathbf{LK}_p^{\eta}$:

$$\dfrac{\begin{array}{cc}\pi_0 & \pi_1 \\ \vdots & \vdots \\ \Gamma, A_s & \Delta, A_{\bar{s}}\end{array}}{\Gamma, \Delta} \quad \textit{and} \quad \dfrac{\begin{array}{cc}\pi_0^r & \pi_1 \\ \vdots & \vdots \\ \Gamma, A_s & \Delta, A_{\bar{s}}\end{array}}{\Gamma, \Delta}$$

where the cutformula A_s is reversible, and $A_{\bar{s}}$ is main in a logical rule. Then both proofs have a common reduct.

PROOF. Note that A_s is non-attractive, but since $A_{\bar{s}}$ is main, it is π_1's turn to move in an S2-step, unless A_s is also main, in which case the two proofs are the same and the lemma holds. Therefore, let us suppose that A_s is *not* main in the left-most derivation. Perform there the S2-step, followed by all the n L-steps on the leaves corresponding to logical introductions in the tree of A_s's ancestors. Let π_{1j} denote the subproof(s) of π_1 corresponding to the premise(s) of π_1's final rule. Then, because of the η-constraint, *if* an S-step is applicable, it will be (some copy of) π_{1j} that moves; perform all these S-steps (at most $2n$ in the multiplicative case and at most n in the additive case). Now turning to the *right-most* proof: perform the L-step to unleash the same subproof(s) π_{1j} of π_1; then perform the following S-steps also. Both ways of reducing lead to the same derivation. ⊣

The perturbation has vanished as in the case of η-expansion in λ-calculus: $(\lambda x\,(M)x)N \to (M)N$. Let us stress that both the η-constraint and the polarization confederate here in order to yield the result.

We now can define an alternative way of 'computing' $\mathbf{LK}^{\eta}$-proofs by substituting a reversion step for the structural step S2 whenever the reversible cutformula is polarized (non-attractive): let's call it the *tqr*-protocol. The lemma above shows that this change in processing cuts doesn't change the normal form: for, it follows by easy diagram-chasing that a proof π and any of π's *tqr*-reducts have a common *tq*-reduct. But might not the new protocol miss that normal form by running eternally?

THEOREM 44. *The tqr-protocol is strongly normalizing.*

PROOF. According to Lemma 42, this new normalization may be simulated by the usual *tq*-protocol through the embedding consisting in plugging the appropriate η wherever a polarized reversible formula is active in a logical rule or a cut. There may be two ways for so doing when two reversible formulas A and B are both active in the same rule, which therefore has to be a unary multiplicative one: in this case any order will do.

When performing a reversion step, one just triggers the reduction of the corresponding cut on the η-proof in the image. When performing an S1 step nothing special happens, neither when performing a logical step except in the special case of a multiplicative logical step where the unary multiplicative rule, say $A \vee_m B$, has both active formulas reversible. Then, after the logical step one gets:

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_{A,B}}{\Longrightarrow A,B} \quad \dfrac{\eta_A}{A \Longrightarrow A}}{\Longrightarrow A,B} \quad \dfrac{\eta_B}{B \Longrightarrow B}}{\Longrightarrow A,B} \quad \dfrac{\pi_A}{A \Longrightarrow}}{\Longrightarrow B} \quad \dfrac{\pi_B}{B \Longrightarrow}}{\Longrightarrow}$$

which needs commuting the two middle cuts to get the desired result. Note that

A in π_A is attractive because its dual A in $\pi_{A,B}$ is reversible and polarized, hence non-attractive. Note also that the same A in π_A was active in an irreversible rule before the logical step. Hence by the η-constraint it was main and still is. Now by Lemma 4, the commutation is harmless. (If η_B is above η_A then just choose the logical step so that π_B goes above π_A and commute the two middle cuts.)

To conclude see that once the main-active interspace of a reversible occurrence of formula is flattened in this way, it remains so, i.e., that a block [reversible rule/cut] where the cutformula is main in the reversible rule is stable by tq-normalization. But this is immediate from (the proof of) Lemma 16 (cf. the remark on page 777).⊣

Consequently this alternative normalization is also confluent. It had already been considered in order to prove cut-elimination for **LU** by Vauzeilles ([45]), and, independently, for **LK** by Joinet ([18]), and probably in many other proof-theoretic inquiries. Presently, the scheme seems to need the structure of formulas due to the expansion steps in the reversion transformation which prevents it from having a 'pure' counterpart.

6.1. Strong equivalence. Two $\mathbf{LK}^{\eta}$-proofs are said to be *strongly equivalent* if they only differ up to reversion of *active or terminal* polarized reversible formulas. (We say a formula is terminal if it occurs in the end sequent.) The proviso that the reversed formula be active makes sure that the η-constraint is preserved by reversion. Indeed, since the formula is non-attractive one may insert a cut with the appropriate η without damaging the constraint on the rule below, and obtain the reversed form by reducing it.

PROPOSITION 45 (strong equivalence is tq-stable). *If two* $\mathbf{LK}^{\eta}$*-proofs are strongly equivalent, then so are their normal forms.*

PROOF. It is equivalent to show that if π' only differs from π up to one active polarized reversion, their normal forms are strongly equivalent. We proceed by induction on the maximal reduction length of π (not necessary, in fact just convenient). If π is normal, then both are their own normal forms, and the proposition holds.

Therefore suppose the reversion takes place above a cut (in which A_s to be reversed is active):

$$\dfrac{\begin{array}{cc}\pi_0 & \pi_1\\ \vdots & \vdots\\ \Gamma, A_s & \Delta, A_{\bar{s}}\end{array}}{\Gamma, \Delta}\ .$$

Then if an L step applies, the proofs are the same. If an S2 step applies, Lemma 43 above says their normal forms are the same. If an S1 step applies then π_0 or π_0' has to climb up $A_{\bar{s}}$'s tree, and once this is done both proofs only differ up to n active polarized reversions, n being the number of axiom and logical leaves in the tree, and hence by induction are strongly equivalent. Note that in the case of an axiom leaf, since the axiom is absorbed, the reversion of A_s stops halfway, so that in this case one also has to reverse the reduct of the reversed proof, but that's ok.

Suppose now the reversion takes place elsewhere, then one has just to reverse residuals of A_s to get the reduct of the reversed proof. ⊣

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Longrightarrow A \qquad B \Longrightarrow B}{A, B \Longrightarrow A \otimes B}}{!A, B \Longrightarrow A \otimes B}}{!A, !B \Longrightarrow A \otimes B}}{A \Longrightarrow !A \qquad !A, !B \Longrightarrow !(A \otimes B)}}{B \Longrightarrow !B \qquad A, !B \Longrightarrow !(A \otimes B)}}{\dfrac{A, B \Longrightarrow !(A \otimes B)}{A \otimes B \Longrightarrow !(A \otimes B)}}$$

FIGURE 6. Being !-fix is closed under $\otimes$.

Now, since strong equivalence is stable, it is tempting to try to 'materialize' the quotient by asking for a fragment of $\mathbf{LK}^{\eta}_{p}$ where active reversible formula occurrences are constrained to be main 'up to reversible rules'.[11] But, while interesting, note that this fragment has to rule out second-order quantification since second order logical steps are sure to violate the constraint.

6.2. The *P*-translation. We now turn to defining a translation in **LL** that goes together well with the strong equivalence.

LEMMA 46. *Let us say that an* **LL***-formula A is* !*-fix (respectively* ?*-fix) if* $A \Longrightarrow !A$ *(respectively* $?A \Longrightarrow A$*). Being* !*-fix (respectively* ?*-fix) is closed under left (respectively right) reversible binary connectives and quantifiers (i.e.,* $\otimes$, $\oplus$ *and* $\exists$ *(respectively* $\wp$, & *and* $\forall$*)).*

As an example, we show in Figure 6 how to prove closure of being !-fix under $\otimes$.

By Lemma 46, the fact that $!A$ (respectively $?A$) is !-fix (respectively ?-fix), and the additional fact that components of right (respectively left) reversible binary connectives and quantifiers get a pre-modalization of type ? (respectively !) in the stringo translation we get:

LEMMA 47. *The S translation of a left (respectively right) reversible binary connective or quantifier is* !*-fix (respectively* ?*-fix).*

This property of the stringo translation in the case of polarized proofs can be turned into a very efficient way to pluck exponentials. Take for instance the stringo translation of a multiplicative implication with t-coloured components, namely $!?S(A) \multimap ?S(B)$. If we suppose that A and B are non-atomic and that the proof within which this implication occurs is polarized, then, since A and B are taken to be t, by definition of polarized proofs, they must be right reversible, hence by the proposition above we know that $S(A)$ is **LL** equivalent to $?S(A)$ and

[11] Added in print: such a syntax has recently been described and investigated by M. Quatrini and L. Tortora de Falco, *Polarisation des preuves classiques et renversement*, to appear in the Comptes Rendus de l'Académie des Sciences.

similarly for $S(B)$. Now this means that we might as well translate the formula by $!S(A) \multimap S(B)$.[12]

More generally, we can go all the stringo way, now using instead of the good old adequate pairs:

$$(\mu_R^t, \mu_L^t) := (?, !?) \qquad (\mu_R^q, \mu_L^q) := (?!, !)$$

their polaro forms (where $\cdot$ means the void modality, as before):

$$(\nu_R^t, \nu_L^t) := (\cdot, !) \qquad (\nu_R^q, \nu_L^q) := (?, \cdot).$$

DEFINITION 48 (polaro translation). We define a mapping of coloured classical formulas to linear formulas by $P(X^t) = ?X$, $P(X^q) = !X$, $P(\neg A) = P(A)^\perp$ and (for $\varepsilon, \varepsilon' \in \{t, q\}$):

$$P(A^\varepsilon \xrightarrow{m} B^{\varepsilon'}) = \nu_L^\varepsilon P(A) \multimap \nu_R^{\varepsilon'} P(B)$$
$$P(A^\varepsilon \xrightarrow{a} B^{\varepsilon'}) = \nu_R^\varepsilon P(A) \rightsquigarrow \nu_L^{\varepsilon'} P(B)$$
$$P(\forall X\, A^\varepsilon) = \forall X\, \nu_R^\varepsilon P(A).$$

Explicitly, P is defined by:

A/B	t	q
t	$!A \multimap B$	$!A \multimap ?B$
q	$A \multimap B$	$A \multimap ?B$

A/B	t	q
t	$A \overset{a}{\rightsquigarrow} !B$	$A \overset{a}{\rightsquigarrow} B$
q	$?A \overset{a}{\rightsquigarrow} !B$	$?A \overset{a}{\rightsquigarrow} B$

Again the same remark: it suffices to consider implications, negation and universal quantifier.

LEMMA 49. *The P-translation of a formula coloured q (respectively t) is !-fix (respectively ?-fix).*

PROOF. A formula is coloured q if and only if it is (1) a left reversible binary connective or quantifier, in which case its P-translation is **LL**-equivalent to its S-translation, hence is !-fix because of Lemma 47; or (2) a q-coloured atom X^q in which case $P(X^q) = !X$ which is indeed !-fix; or (3) the negation of a t-coloured formula, in which case its P-translation is the **LL**-negation of a ?-fix formula (here again the flipping colours condition shows up). ⊣

PROPOSITION 50 (polaro embedding). *The translation of Definition 48 allows for an inductive mapping of* $\mathbf{LK}_p^\eta$ *proofs (even of polarized proofs of* $\mathbf{LK}^{tq}$*) to proofs of* **LL**.

PROOF. A proof of a sequent $\Gamma^t, \Lambda^q \Longrightarrow \Delta^t, \Sigma^q$ is now mapped to a proof of $!P(\Gamma), P(\Lambda) \Longrightarrow P(\Delta), ?P(\Sigma)$ (instead of $!?S(\Gamma), !S(\Lambda) \Longrightarrow ?S(\Delta), ?!S(\Sigma)$ as in the simply stringent embedding). Structural rules and promotions, whenever necessary, will use cuts on the **LL**-proofs implementing !- or ?-fixness properties of Lemma 49. Reversible and negation rules are now translated with no modalization at all (neither

[12] Were A and B left reversible, as in $\mathbf{LK}_{ap}^\eta$, then their !-fix property would be useless; in this case we would seek for a !?-fix property for left reversible connectives (like: $!?(!?X \otimes !?Y)$ **LL**-equivalent to $!?X \otimes !?Y$, which is indeed the case) to get the 'sparse' antipolaro translation $S(A) \multimap ?S(B)$.

pre- nor post-), while irreversible rules are translated with a promotion on non-attractive active formulas (pre-modalization) and a dereliction on the main formula (post-modalization). ⊣

Note that this embedding is far from being a decoration. Also, the polaro translation is not substitutive: were we to consider second order reduction steps, then formulas would appear that are no longer of the P-form.

§7. FD, LC and their relatives. In this section we will sketch the relation between $\mathbf{LK}^{tq}$, $\mathbf{LK}^{\eta}$, $\mathbf{LK}^{\eta}_{p}$, and other constructive proof systems for classical logic, namely and respectively **FD** ([32]), $\lambda\mu$ ([33]), **LC** ([10]).

Within unrestricted $\mathbf{LK}^{tq}$ we recover Parigot's Free Deduction **FD**. More precisely, the obvious mapping from **FD** into (heterostyle) $\mathbf{LK}^{tq}$, which consists in reading **FD**-rules as cuts and 'input left' (respectively 'input right') as t (respectively q), is a homomorphism, i.e., **FD**-normalization steps are simulable through this mapping by means of tq-steps. Hence strong normalization for **FD** becomes a corollary to Theorem 10. Technical details are worked out in [19]. Note that **FD**'s binary rules (corresponding to unary **LK** rules) are taken *additive* as for instance:

$$\frac{\Longrightarrow B \quad A \to B \Longrightarrow}{\Longrightarrow}, \qquad \frac{A \Longrightarrow \quad A \to B \Longrightarrow}{\Longrightarrow},$$

so as to escape the logical dilemma.

Parigot then obtained his classical natural deduction (or $\lambda\mu$-calculus) from **FD** by restricting proofs, like we did when passing to $\mathbf{LK}^{\eta}$. In his case the constraint consists in demanding that certain premises in the **FD**-rules be *axioms*. Just the same homomorphism that maps **FD** into $\mathbf{LK}^{tq}$ maps his classical natural deduction into **LKT**, that is the fragment of $\mathbf{LK}^{\eta}$ where all formulas are t (because all formulas are asked to have input to the left). Note that in the restricted system heterostyle connectives are no longer needed and one may safely return to the good old multiplicative introduction rule

$$\frac{A \Longrightarrow B}{\Longrightarrow A \to B}.$$

Indeed this is what happens in [33].

LKT and **LKQ** defined in [18], [40], [4] are, as said, the fragments of $\mathbf{LK}^{\eta}$ where all formulas are t, respectively q. These systems, however, as Girard's **LC** and **LU**, are formulated as calculi in which sequents contain, possibly, some *distinguished* formula-occurrence: the formula in 'the stoup', separated from other formulas by means of a ";", as in $A; \Gamma \Longrightarrow \Delta$, for example. This syntactic trick allows the prescription in rules of constraints bearing on main-active interspaces. One might in fact formulate $\mathbf{LK}^{\eta}$ in a similar way, and, historically, that is how it was done. But observe that in order to formulate $\mathbf{LK}^{\eta}$ explicitly, one has to consider all possible colourings of the active formulas and the main formula in a given rule, which for each binary connective (multiplicative + additive) amounts to no less than $(2+3)\times 2^3 = 40$ distinct rules! (Cf. also [14] for more numerological consequences.) One will surely prefer the elegant Definition 20.

Colours t and q corresponds respectively to 'negative' and 'positive' in **LC**'s terminology. This said, **LC**, up to the stoup/no-stoup formulation, is but a fragment

of $\mathbf{LK}^{\eta}_{p}$, where an additional constraint ties the style of a given connective and the colours of its subformulas. For instance the only case where the additive implication may be used is $A^{t} \xrightarrow{a} B^{q}$, whereas all other cases ask for the multiplicative implication.

Of course we here have as a consequence of Theorem 10 that the restriction of the tq-normalization to **LC** is strongly normalizing and confluent. Note also that our analysis of the tqr variant of normalization solves the riddle whether **LC** should be normalized the tq-way as the syntactic part of Girard's paper suggests, or the tqr-way as the semantic part does (Vauzeilles takes this option in [45]).

§8. Intuitionistic logic: a 'would-be' deconstructive logic? Retrospectively, we *can* reconstruct certain parts of the above treatment of the proof theory of classical logic directly within intuitionistic logic, that former champion of constructivization and constructivism. In doing so, moreover, **LJ**'s defects as a 'classifying and clarifying tool', will clearly come to the fore.

From the point of view of reduction of sequent calculus proofs, the 'control regained' in the 'constructive' intuitionistic calculus is based upon a brute dissymmetrisation: the use of structural rules, especially of *contraction*, on the rhs is forbidden. The well-known embeddings of classical into intuitionistic logic using Gödel-Gentzen-Kolmogorov-like negative translations (cf. [44]) in fact can be interpreted as systematic ways of replacing rhs applications of structural rules on a formula A by lhs applications of (the same) structural rules on its negation $\neg A$.

This interpretation of negative translations immediately leads to a notion of 'intuitionistic' or 'negative' (inductive) decoration of **LK**-derivations, in which **LK**-sequents $\Gamma \Longrightarrow \Delta$ become $\Gamma^{\circ}, \neg\Delta^{\circ} \Longrightarrow$ (where $(\cdot)^{\circ}$ indicates an inductive translation of formulas, yet to be determined).

What becomes of an instance of a cut rule

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2 \\ \vdots & \vdots \\ \Longrightarrow A & A \Longrightarrow\end{array}}{\Longrightarrow}$$

when we negatively decorate an **LK**-derivation?

By induction hypothesis, the premises $\Longrightarrow A$ and $A \Longrightarrow$, have been interpreted as $\neg A^{\circ} \Longrightarrow$, respectively $A^{\circ} \Longrightarrow$.

Hence we complete our derivation in **LJ** as follows:

$$\dfrac{\begin{array}{cc}\pi_1^{\circ} & \pi_2^{\circ} \\ \vdots & \vdots \\ \neg A^{\circ} \Longrightarrow & A^{\circ} \Longrightarrow \\ & \Longrightarrow \neg A^{\circ}\end{array}}{\Longrightarrow}.$$

Consequently, each instance of cut, when negatively decorated, becomes an instance of intuitionistic cut in which the positive occurrence of the cutformula is main formula in a negation-rule, i.e., in a logical rule. (Cf. the linear decorations,

in which each instance of a cut becomes an instance of linear cut in which one of the cutformulas is main in an exponential contextual rule, i.e., is of type '!'.) Hence, if we consider *tq*-normalization of **LJ**-derivations, in case the left cutformula is *not* main in a negation rule, π_2° will be the transported subderivation. So we see that, after a negative translation, the structural steps are independent of the colour of the formulas, and, by pullback, correspond to a *q*-type normalization of the original **LK**-derivation. (Indeed, for if the left cutformula $(\neg A^\circ \Longrightarrow)$ is *main* in a negation rule, this is because the positive occurence of the cutformula in the original **LK**-derivation $(\Longrightarrow A)$ is main.[13])

But clearly the dissymmetrisation imposed by **LJ** is terribly ad hoc, and not obviously better than the dual dissymmetrisation corresponding to the logic in which the *lhs* of the classical sequents are restricted, a logic that might be deemed *anti-*, or better: *co-intuitionistic* and in fact, in an AI-context, has been studied in [16]. There it is called **LP** and we will stick to that name.

Classical logic is embeddable in **LP** by what we might call 'positive' translations, which can be interpreted as systematic ways of replacing lhs applications of structural rules on a formula A by rhs applications of (the same) structural rules on its negation $\neg A$. As before, we now obtain a notion of 'co-intuitionistic' or 'positive' (inductive) decoration of **LK**-derivations, in which **LK**-sequents $\Gamma \Longrightarrow \Delta$ become $\Longrightarrow \neg\Gamma^\flat, \Delta^\flat$ (where $(\cdot)^\flat$ indicates an inductive translation of formulas, to be determined later on).

Having seen the discussion above, it will come as no great surprise that the method of pulling back the (*tq*-)normalization of positively decorated derivations naturally leads to *t*-type normalization of classical proofs. We leave verification to the reader. As the 'negative' decoration, the 'positive' decoration has the property that its *tq*-normalization does not depend on colours, i.e., *t*-reducing = *q*-reducing.

Morally, the situation is as in Figure 7, where the arrow from **LK** to **LJ** represents the *negative* decoration of **LK**-derivations, mapping sequents $\Gamma \Longrightarrow \Delta$ to intuitionistic sequents $\Gamma^\circ, \neg\Delta^\circ \Longrightarrow$, that from **LK** to **LP** the *positive* decoration, which maps a sequent $\Gamma \Longrightarrow \Delta$ to the co-intuitionistic $\Longrightarrow \neg\Gamma^\flat, \Delta^\flat$. The arrows from **LJ** and **LP** to **LL** stand for the intuitionistic (respectively co-intuitionistic) pletho-decoration, which sends sequents $\Gamma \Longrightarrow A$ (respectively $A \Longrightarrow \Delta$) to sequents $!\Gamma^\star \Longrightarrow !A^\star$ (respectively $?A^* \Longrightarrow ?\Delta^*$).

Though the above interpretation seems the most natural one, e.g., in the sense that it uses as little negations as possible, it is not *necessary* to use both **LJ** and **LP** in order to find 'all of *tq*'. For, as an **LP**-derivation of $\Longrightarrow \neg\Gamma^\flat, \Delta^\flat$ is of course an **LK**-derivation of that sequent, we can apply a 'negative' decoration to it, and obtain an **LJ**-derivation of $\neg\neg\Gamma', \neg\Delta' \Longrightarrow$. Whence, by 'transitivity of pullbacks', *tq*-normalizing this **LJ**-derivation corresponds to *t*-normalizing the underlying classical proof!

Similarly, if one applies a 'positive' decoration to the 'negative' decoration of an **LK**-derivation of $\Gamma \Longrightarrow \Delta$ will lead to an **LP**-derivation of $\Longrightarrow \neg\Gamma', \neg\neg\Delta'$, whose *tq*-normalization induces the *q*-normalization of the underlying classical proof.

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[13] When decorating, in such a case we may skip the negation steps and cut directly on A°. Cf. the *adjusted* pletho-decoration in Section 4.3.

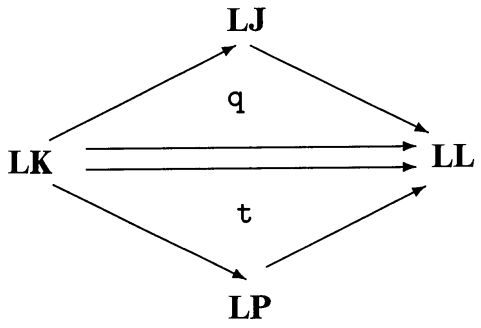

FIGURE 7. Decorations and normalizations.

It is immediate by the above observations, that in both **LJ** and **LP** we can simulate the tq-normalization of $\mathbf{LK}^{tq}$-derivations, by interpreting sequents $\Gamma^t, \Lambda^q \Longrightarrow \Delta^t, \Sigma^q$ by sequents $\neg\neg\Gamma^\circ, \Lambda^\circ, \neg\Delta^\circ, \neg\Sigma^\circ \Longrightarrow$ in the first, by $\Longrightarrow \neg\Gamma^\flat, \neg\Lambda^\flat, \Delta^\flat, \neg\neg\Sigma^\flat$ in the second case. As a consequence we can simulate the tq-normalization of $\mathbf{LK}^{tq}$-derivations in both $\mathbf{LK}^q$ and $\mathbf{LK}^t$, so that no real new algorithms lurk in the bicoloured universe.

Also, except for those with a strong inclination to the left, there is no formal reason to prefer the intuitionistic over the co-intuitionistic solution. Either dissymmetrisation in fact blurs our view ('aberration optique') when analysing the properties of classical cut elimination: when turning to the, up till here skipped over, translation of formulas, it is obvious that the dissymmetrisation makes a 'connective-preserving' translation impossible, as **LJ** lacks a multiplicative disjunction and **LP** lacks a multiplicative conjunction.

The following translation $(\cdot)^\circ$ of (multiplicative) tq-classical formulas is induced by the negative decoration of $\mathbf{LK}^{tq}$-derivations and the additional demand that the number of negations introduced be as small as possible.

$A \overset{m}{\rightarrow} B$	t	q
t	$\neg\neg A \supset \neg\neg B$	$\neg\neg A \supset \neg\neg B$
q	$\neg\neg A \supset \neg\neg B$	$\neg B \supset \neg A$

$A \wedge_m B$	t	q
t	$\neg\neg A \wedge_m \neg\neg B$	$\neg\neg A \wedge_m \neg\neg B$
q	$\neg\neg A \wedge_m \neg\neg B$	$\neg\neg A \wedge_m \neg\neg B$

$A \vee_m B$	t	q
t	$\neg\neg\neg A \supset \neg\neg B$ $\neg\neg\neg B \supset \neg\neg A$	$\neg B \supset \neg\neg A$
q	$\neg A \supset \neg\neg B$	$\neg A \supset \neg\neg B$ $\neg B \supset \neg\neg A$

('$\supset$' stands for the (multiplicative) intuitionistic implication.)

$\neg A$	
t	$\neg\neg\neg A$
q	$\neg A$

We leave it to the reader to construct the similar translation $(\cdot)^b$ induced by the positive decoration of $\mathbf{LK}^{tq}$-derivations.

As an interesting illustration of how the dissymmetrisation imposed by **LJ**, **LP** stands in the way of a clear analysis of classical cut-elimination, observe that the negative decoration of the unary rule for $A^q \xrightarrow{m} B^t$ is *unique*. In fact, the negative decoration imposes the orientation 'A-cut before B-cut'. Similarly, the positive decoration imposes the orientation 'B-cut before A-cut'. However, the critical cases for the other binary connectives *do* allow for two distinct decorations of the unary rule, corresponding to the two possible orientations, as in our linear analysis. Cf. page 768.

The translation $\neg\neg A \supset \neg\neg B$ with sequents translated as $\neg\neg\Gamma, \neg\Delta \Longrightarrow$ was independently considered by Parigot in [31] to prove strong normalization for his classical natural deduction by exhibiting a homomorphism to intuitionistic natural deduction. Murthy in [28, 29] applied similar ideas to "control calculi" (that is λ-calculi extended with exception mechanisms, quite close to $\lambda\mu$-calculus). These calculi antedate the proof-theoretic investigations of the normalization of classical logic and were recognized to be typable in propositional classical logic by Griffin in [13]. At the level of (extended) λ-terms those translations induce so-called continuation-passing-style compilations into λ-terms. Such CPS translations were first defined by Plotkin in [37].

§9. *LK* as a programming language. This last section considers **LK** as a programming language, inasmuch as it provides a way to extend the Krivine/Leivant programming method from $\mathbf{LJ}^2$ to $\mathbf{LK}^2$. It is kind of a case study in applied proof theory that, *in fine*, justifies the energy we invested in formulating a normalization for **LK**. The programming method is explained in [26, 34] and independently was considered by Leivant in [27]. A program consists in a 'totality proof' that is a proof that an equational system E defines a total function f. This proof is shown to always be an implementation of the equational system, which roughly means that cut with (a proof representing) an integer n, the totality proof will normalize to (a proof representing) an integer m such that $f(n) = m$ is derivable in E. However in **LK** it needs a special device (a so-called converter) to know *which* integer is represented by a given proof. We will give a precise formulation of the "programming theorem" at the end of this section. First, we need to add equational reasoning to $\mathbf{LK}^2$ and **LL**. Let an equational theory E be given, then if $t = t'$ holds in E we allow such rules as:

$$\frac{\Gamma \Longrightarrow \Delta, A(t)}{\Gamma \Longrightarrow \Delta, A(t')} \qquad \frac{\Gamma, A(t) \Longrightarrow \Delta}{\Gamma, A(t') \Longrightarrow \Delta.}$$

Note that there is an obvious extension of tq-normalization to this extended calculus; also the strong normalization theorem evidently continues to hold. Second, we need *classical integers*:

$$N(x) := \forall X\,((\forall z\,(X(z) \to X(Sz))) \to (X(0) \to X(x))),$$

this is the usual second order definition of integers (we take the implication's style to be multiplicative, colours are as you wish); as well as *linear integers*:

$$N'(x) := \forall X\,((!(\forall z\,(X(z) \multimap X(Sz)))) \multimap (X(0) \multimap X(x))),$$

Up to the permutations of left implication rules and weak equivalence, there is a unique normal proof without equational reasoning, of $\Longrightarrow N'(S^n0)$ which is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X(0) \Longrightarrow X(0) \qquad X(S0) \Longrightarrow X(S0) \qquad \ldots \qquad X(S^n0) \Longrightarrow X(S^n0)}{X(0), X(0) \multimap X(S0), \ldots, X(S^{n-1}0) \multimap X(S^n0) \Longrightarrow X(S^n0)}}{X(0), \forall z\,(X(z) \multimap X(Sz)), \ldots, \forall z\,(X(z) \multimap X(Sz)) \Longrightarrow X(S^n0)}}{X(0), !\forall z\,(X(z) \multimap X(Sz)) \Longrightarrow X(S^n0)}}{\Longrightarrow (!\forall z\,(X(z) \multimap X(Sz))) \multimap (X(0) \multimap X(S^n0))}}{\Longrightarrow N'(S^n0)}$$

Now let π be a normal derivation *with* equational reasoning of $\Longrightarrow N'(t)$ and let $\sharp\pi$ denote the number of axioms in π minus one; then π "contains" an equational proof of $t = S^{\sharp\pi}0$. Giving a precise meaning to this, a task which we leave to the reader, is merely a matter of describing all ways one can possibly add equational reasoning to a proof such as the unique proof displayed above.

A slight extension of that property, which will be what we actually use in the sequel, is: any normal proof π of $\Longrightarrow ?N'(t)$ ends, modulo contracted weakenings, with an instance of *dereliction* and hence also contains an equational proof of $t = S^{\sharp\pi}0$.

Indeed, for any normal proof net proving $?\forall X\,F$ it is easily seen that two atoms X and $X^\perp$ above two different premises of the conclusion $?\forall X\,F$ cannot be conclusions of the same axiom node, for if they do then their respective forall's, the ones they are above, might be linked in a correction graph one to the other (because their premise contain the atom which the other one binds, in other words eigenvariables are the same) and this produces a cycle (we use the correctness criterion extended to quantifiers as presented in [11]). Then, provided that F has no predicate variables other than X, any such proof net has to be a contraction on some number, say p, of proof structures with only conclusion $\forall X\,F$, each of them with acyclic correction graphs. Now any normal proof structure with acyclic correction graphs and with N' as only conclusion is in fact a proof net, i.e., comes from sequent calculus. So $p = 1$ for any normal proof net proving $?N'$, and any of its sequent versions has to end with a dereliction (would the proof net only come from sequent calculus augmented with the juxtaposition rule:

$$\dfrac{\Gamma \quad \Delta}{\Gamma, \Delta},$$

then it could be mapped back to that augmented sequent calculus as a juxtaposition of $p > 1$ sequent proofs of $\Longrightarrow N'$ followed by $p - 1$ contractions; a remark that we will use at the very, very end of the paper).

We call τ a *totality proof* for f if it proves $N(x) \Longrightarrow N(f(x))$; we say that χ is a *converter* if it is a skeleton of some **LL**-proof of $D(N(x)) \Longrightarrow ?N'(x)$ (where

D stands for the pletho decoration). (Note that there is some undeterminism in the skeleton operation since one has to choose colours on the way back to **LK**.)

Say a converter is *standard* if the **LL**-proof of which it is a skeleton is a strip of its pletho-decoration, i.e., differs from this decoration only up to superfluous exponentials (cf. [3]).

We now have to make sure that converters exist. Let $(\Box_i)$ denote the modalities used in $D(N(x))$. The induction scheme associated to that species of integer is:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\overset{\text{step}}{\vdots}}{Z(z) \Longrightarrow Z(Sz)}
}{\Longrightarrow \Box_2(\Box_3 Z(z) \multimap \Box_4 Z(Sz))}
}{\Longrightarrow \Box_1 \forall z\, \Box_2(\Box_3 Z(z) \multimap \Box_4 Z(Sz))}
\qquad
\dfrac{
\dfrac{
\dfrac{\overset{\text{base}}{\vdots}}{\Longrightarrow Z(0)}
}{\Longrightarrow \Box_6 Z(0)}
\qquad \Box_7 Z(x) \Longrightarrow \Box_8 Z(x)
}{\Box_6 Z(0) \multimap \Box_7 Z(x) \Longrightarrow \Box_8 Z(x)}
}{\Box_5(\Box_6 Z(0) \multimap \Box_7 Z(x)) \Longrightarrow \Box_8 Z(x)}
}{(\Box_1 \forall z\, \Box_2(\Box_3 Z(z) \multimap \Box_4 Z(Sz))) \multimap \Box_5(\Box_6 Z(0) \multimap \Box_7 Z(x)) \Longrightarrow \Box_8 Z(x)}
}{\forall X\, \Box_0 \{(\Box_1 \forall z\, \Box_2(\Box_3 X(z) \multimap \Box_4 X(Sz))) \multimap \Box_5(\Box_6 X(0) \multimap \Box_7 X(x))\} \Longrightarrow \Box_8 Z(x)}
}{D(N(x)) \Longrightarrow \Box_8 Z(x)}
$$

where it is enough to suppose $\Box_8$ of type ? when $\Box_0$ and $\Box_5$ are so.

With $\Box_8 = ?$, $Z = N'$, step:

$$
\dfrac{
\dfrac{
\dfrac{
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\dfrac{
\dfrac{
\dfrac{!(\forall z\,(X(z) \multimap X(Sz))) \Longrightarrow !(\forall z\,(X(z) \multimap X(Sz))) \qquad X(0) \Longrightarrow X(0) \qquad X(x) \Longrightarrow X(x)}{(!(\forall z\,(X(z) \multimap X(Sz)))) \multimap (X(0) \multimap X(x)), !(\forall z\,(X(z) \multimap X(Sz))), X(0) \Longrightarrow X(x)}
}{N'(x), !(\forall z\,(X(z) \multimap X(Sz))), X(0) \Longrightarrow X(x)}
\qquad X(Sx) \Longrightarrow X(Sx)
}{N'(x), !(\forall z\,(X(z) \multimap X(Sz))), X(0), X(x) \multimap X(Sx) \Longrightarrow X(Sx)}
}{N'(x), !(\forall z\,(X(z) \multimap X(Sz))), X(0), !(\forall z\,(X(z) \multimap X(Sz))) \Longrightarrow X(Sx)}
}{N'(x), !(\forall z\,(X(z) \multimap X(Sz))), X(0) \Longrightarrow X(Sx)}
}{N'(x) \Longrightarrow (!(\forall z\,(X(z) \multimap X(Sz)))) \multimap (X(0) \multimap X(Sx))}
}{N'(x) \Longrightarrow N'(Sx)}
$$

where exponentials are introduced on main formulas only, which is enough to be a strip of some decoration of the skeleton, and base:

$$
\dfrac{
\dfrac{
\dfrac{X(0) \Longrightarrow X(0)}{!(\forall z\,(X(z) \multimap X(Sz))), X(0) \Longrightarrow X(0)}
}{\Longrightarrow (!(\forall z\,(X(z) \multimap X(Sz)))) \multimap (X(0) \multimap X(0))}
}{\Longrightarrow N'(0)}
$$

one gets a proof of $D(N(x)) \Longrightarrow ?N'(x)$, the skeleton of which is a converter, even a standard one.

THEOREM 51 (programming theorem). *Let τ be a totality proof for f, χ a standard converter and ω a proof of $\Longrightarrow N(t)$. Then the normal form π_0 of:*

$$\dfrac{\begin{array}{ccc}\omega & \tau & \chi \\ \vdots & \vdots & \vdots \\ \Longrightarrow N(t) & N(x) \Longrightarrow N(f(x)) & N(f(x)) \Longrightarrow N(f(x))\end{array}}{\Longrightarrow N(f(t))}$$

contains an equational proof of $S^{\sharp\pi_0}0 = f(t)$.

PROOF. The adjusted pletho-decoration $D^-(\pi)$ of $\pi = \mathrm{cut}(\omega, \tau, \chi)$ strips to a proof π' of $\Longrightarrow ?N'(f(t))$, since χ is standard. By the simulation theorem, Theorem 9, any normalization of π to π_0 is simulated by some normalization of $D^-(\pi)$ to the normal form $D^-(\pi_0) = D(\pi_0)$ which, because normalization preserves stripping, strips to the normal form of π'. Then π_0 is a skeleton of that